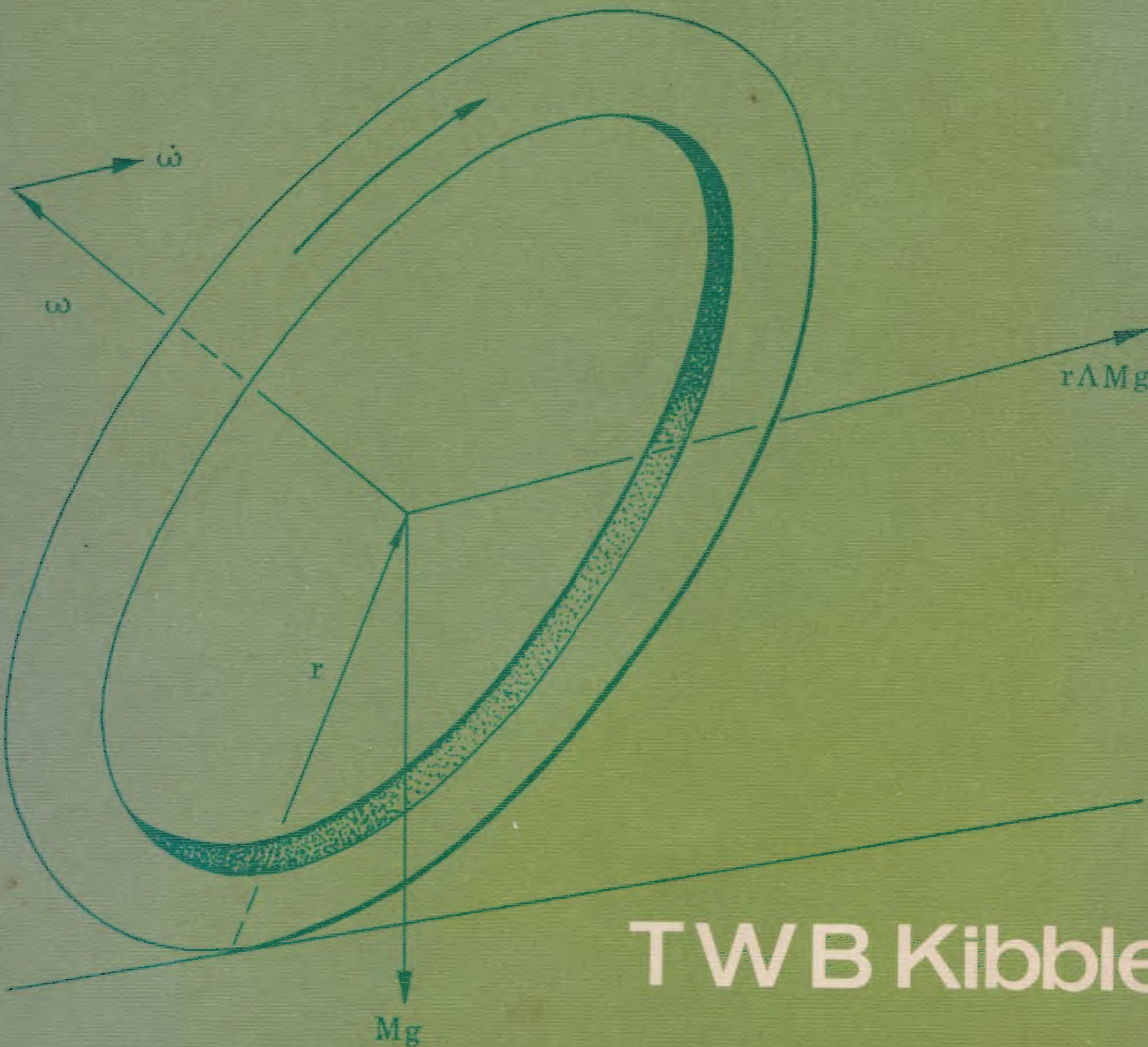
Second Edition

Classical Mechanics



T W B Kibble

List of Symbols

The following list is not intended to be exhaustive. It includes symbols of frequent occurrence or special importance. The figures refer to the section in which the symbol is defined.

A, A_α	complex amplitude	2.3, 12.3
$\mathbf{A}$	vector potential	B
$\mathbf{a}$, a	acceleration	1.2
a	amplitude of oscillation	2.2
a	semi-major axis of orbit	4.4
$\mathbf{B}$	magnetic field	B
b	semi-minor axis of orbit	4.4
b	impact parameter	4.3, 4.5
c	velocity of light	D
c	propagation velocity	11.6
$\mathbf{d}$	dipole moment	6.2
E	total energy	2.1, 3.1, 8.5
$\mathbf{E}$	electric field	6.1, B
e	base of natural logarithms	
e	electronic charge	D
e	eccentricity of orbit	4.4
$\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$	principal axes	9.4, C.3
$\mathbf{F}$, $\mathbf{F}_i$, $\mathbf{F}_{ij}$	force	1.2, 8.1
F_α	generalized force	3.6, 11.2
f	flux	4.5
G	gravitational constant	1.2
$\mathbf{G}$	moment of force	3.2
g, $\mathbf{g}$	gravitational acceleration	5.3, 6.1
$\mathbf{g}^*$	observed gravitational acceleration	5.3
H	Hamiltonian function	13.1
I	action integral	3.6, 11.2
I, I_{xx}	moment of inertia	9.2
I_{xy}	product of inertia	9.3
I_i, I_i^*	principal moments of inertia	9.4, 9.5
i	$\sqrt{(-1)}$	
$\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$	unit vectors	A.1
$\mathbf{J}$, $\mathbf{J}^*$	angular momentum	3.2, 7.1, 8.3
k, $k_{\alpha\beta}$	oscillator constant	2.2, 12.2
k	inverse square law constant	4.3
L	Lagrangian function	3.6, 11.2
l	semi-latus rectum of orbit	4.3, 4.4
ln	natural logarithm	
M	total mass	7.1, 8.1
m, m_i	mass	1.2
N	number of particles	1.2
n	number of degrees of freedom	11.1
$\mathbf{P}$	total momentum	7.1, 8.1

List of Symbols

$\mathbf{p}$, $\mathbf{p}_i$, $\mathbf{p}^*$	momentum	1.2, 7.2
p_α	generalized momentum	3.6, 13.1
p	exponential factor	2.2
Q	quadrupole moment	6.2
Q	quality factor	2.5
q, q_i	charge	1.2, 6.1
q_i	curvilinear co-ordinate	3.6
q_α	generalized co-ordinate	11.1
$\mathbf{R}$	position of centre of mass	7.1, 8.1
r	radial co-ordinate	3.4
$\mathbf{r}$, $\mathbf{r}_i$	position vector	A.1, 1.1
$\mathbf{r}$, $\mathbf{r}_{ij}$	relative position	7.1, 1.2
$\mathbf{r}_i^*$	position in CM frame	7.2, 8.3
T, T^*	kinetic energy	2.1, 7.2, 8.5
t	time	1.1
U	effective potential energy function	4.2, 13.4
V	potential energy	2.1, 3.1, 8.5
$\mathbf{v}$, $\mathbf{v}_i$	velocity	1.2
$\mathrm{d}W$, δW	work done in small displacement	2.4, 11.2
X, Y, Z	co-ordinates of centre of mass $\mathbf{R}$	11.1
x, y, z	Cartesian co-ordinates	A.1
α, γ	damping constants	2.5
δ	small variation	3.5
θ	polar angle, Euler angle	3.4, 10.4
Θ, θ, θ^*	scattering angle	4.4, 7.3
μ	reduced mass	7.1
ρ	cylindrical polar co-ordinate	3.4
ρ	mass or charge density	6.3
σ, $\mathrm{d}\sigma$	cross-section	4.5
τ	period	2.2, 4.4
Φ	gravitational potential	6.1
ϕ	electrostatic potential	6.1, B
φ	azimuth angle, Euler angle	3.4, 10.4
ψ	Euler angle	10.4
ω	angular frequency	2.2
$\boldsymbol{\omega}$, $\boldsymbol{\Omega}$	angular velocity	5.1, 10.2
$\mathrm{d}\Omega$	solid angle	4.5
$\boldsymbol{\nabla}$	vector differential operator	A.4

Classical Mechanics

Second edition

European Physics Series

Consulting Editor:

Professor J. M. Cassels, M.A., Ph.D., F.R.S.
Lyon Jones Professor of Physics
University of Liverpool

Other titles in the series:

Adkins: Equilibrium Thermodynamics
Braddick: Vibrations, Waves and Diffraction
Cassels: Basic Quantum Mechanics
Duffin: Electricity and Magnetism, Second Edition
Duffin: Advanced Electricity and Magnetism
Elton: Concepts of Classical Mechanics
Greig: Electrons in Metals and Semiconductors
Jackson: Concepts of Atomic Physics
Matthews: Introduction to Quantum Mechanics, Second Edition
Mattuck: A Guide to Feynman Diagrams in the Many-Body Problem
Squires: Practical Physics
Woodgate: Elementary Atomic Structure

Classical Mechanics

Second edition

T.W.B.Kibble

Professor of Theoretical Physics,
Imperial College of Science and Technology, London

London · New York · St Louis · San Francisco ·
Düsseldorf · Johannesburg · Kuala Lumpur ·
Mexico · Montreal · New Delhi · Panama · Paris ·
São Paulo · Singapore · Sydney · Toronto

Published by
McGRAW-HILL Book Company (UK) Limited
MAIDENHEAD · BERKSHIRE · ENGLAND

Hard cover 07 084018 0
Soft cover 07 084019 9

PRINTED AND BOUND IN GREAT BRITAIN

Contents

To ANNE

Preface

This book, based on a course given to physics students at Imperial College, deals with the mechanics of particles and rigid bodies. It is designed for students with some previous acquaintance with the elementary concepts of mechanics, but the book starts from first principles, and little detailed knowledge is assumed. An essential prerequisite is a reasonable familiarity with differential and integral calculus, including partial differentiation. However, no prior knowledge of differential equations should be needed. Vectors are used from the start in the text itself; the necessary definitions and mathematical results are collected in an appendix.

Classical mechanics lacks the glamour, which attaches to quantum mechanics or relativity, of being in the forefront of modern physics. Its practical utility is obvious, but it often tends to be regarded as a rather unexciting subject. In fact, however, many of the most fascinating recent discoveries about the nature of the earth and its surroundings—particularly since the launching of artificial satellites—are direct applications of classical mechanics. Several of these are discussed in the following chapters. For physicists, though, the real importance of classical mechanics lies not so much in the vast range of its applications as in its role as the base on which the whole pyramid of modern physics has been erected. This book, therefore, emphasizes those aspects of the subject which are of importance in quantum mechanics and relativity—particularly the conservation laws, which in one form or another play a central role in all physical theories.

The first five chapters are primarily concerned with the mechanics of a single particle, and Chapter 6, which could be omitted without substantially affecting the remaining chapters, deals with potential theory. Systems of particles are discussed in Chapters 7 and 8, and rigid bodies in Chapters 9 and 10. The powerful methods of Lagrange are introduced at an early stage, and in simple contexts, and developed more fully in the last three chapters of the book. The final chapter contains a discussion of Hamiltonian mechanics, and particularly of the relationship between symmetries and conservation laws—a subject directly relevant to the most modern developments of physics.

I am indebted to several colleagues for valuable suggestions, and particularly to Professor P. T. Matthews for providing encouragement and helpful advice throughout the writing of the book.

The most significant change in this revised edition is probably the

inclusion of answers to the problems, in response to many requests. I have also taken the opportunity to improve, or bring up to date, the presentation of some sections, and to correct a number of errors. I am grateful to several readers for pointing these out.

Imperial College. T. W. B. Kibble

Introduction

Chapter 1

Classical mechanics is one of the most familiar of scientific theories. Its basic concepts—mass, acceleration, force, and so on—have become very much a part of our everyday modes of thought. Thus there is a danger that we may regard their physical meaning as more obvious than it really is. For this reason, a large part of this introductory chapter will be devoted to a critical examination of the fundamental concepts and principles of mechanics.

Every scientific theory starts from a set of hypotheses, which are suggested by our observations, but represent an idealization of them. The theory is then tested by checking the predictions deduced from these hypotheses against experiment. If many such tests are made, and no disagreement is found, then the hypotheses gradually acquire the status of 'laws of nature'. It must be remembered however that, no matter how impressive the evidence may be, we can never claim for these laws a universal validity. We may only be confident that they provide a good description of that class of phenomena for which their predictions have been adequately tested. One of the earliest examples is provided by Euclid's axioms. On any ordinary scale, they are unquestionably valid, but we are not entitled to assume that they should necessarily apply either on a cosmological or a sub-microscopic scale. Indeed, they have been modified in Einstein's theory of gravitation ('general relativity').

validity of physical hypothesis

range of validity w.r.t measure of distance and relative scale.

The laws of classical mechanics are no exception. Since they were first formulated by Newton, their range of known validity has been enormously extended, but in two directions they have been found to be inadequate. For the description of the small-scale phenomena of atomic and nuclear physics, classical mechanics has been superseded by quantum mechanics; and, for phenomena involving speeds approaching that of light, by relativity. Nevertheless, within its range of validity, classical mechanics is a remarkably successful theory, which provides a coherent and satisfying description of phenomena as diverse as the planetary orbits, the tides and the motion of a gyroscope. Moreover, even outside this range, many of the results of classical mechanics still apply. In particular, the conservation laws of energy, momentum and angular momentum are, so far as we yet know, of universal validity.

1.1 Space and Time

The most fundamental assumptions of physics are probably those §1.1.A
concerned with the concepts of space and time. We assume that

space and time are continuous, that it is meaningful to say that an event occurred at a specific point in space and a specific instant of time, and that there are universal standards of length and time (in the sense that observers in different places and at different times can make meaningful comparisons of their measurements). These assumptions are common to the whole of physics, and there is no convincing evidence that we have as yet reached the limits of their range of validity.

In 'classical' physics, we assume further that there is a universal time scale (in the sense that two observers who have synchronized their clocks will always agree about the time of any event), that the geometry of space is Euclidean, and that there is no limit in principle to the accuracy with which we can measure positions and velocities. These assumptions have been somewhat modified in quantum mechanics and relativity. Here, however, we shall take them for granted, and concentrate our attention on the more specific assumptions of classical mechanics.

§1.1. B *The Relativity Principle.* In ancient Greek conceptions of the universe, the fact that heavy bodies fall downwards was often explained by supposing that each element (earth, air, fire, water) has its own appointed sphere, to which it tends to return unless forcibly prevented from so doing. Earth, in particular, tends to get as close as possible to the centre of the universe, and therefore forms a sphere about this point. In this kind of explanation, the central point plays a special, distinguished role, and position in space has an absolute meaning.

In Newtonian mechanics, on the other hand, bodies fall downwards because they are attracted towards the *earth*, rather than towards some fixed point in space. Thus position has a meaning only relative to the earth, or to some other body. In just the same way, velocity has only a relative significance. Given two bodies moving with uniform relative velocity, it is impossible in principle to decide which of them is at rest, and which moving. This statement, which is of fundamental importance, is the *principle of relativity.*

Acceleration, however, still retains an absolute meaning, since it is experimentally possible to distinguish between motion with uniform velocity and accelerated motion. If we are sitting inside an aircraft, we can easily detect its acceleration, but we cannot measure its velocity—though by looking out we can measure its velocity *relative* to objects outside.*

* In Einstein's theory of gravitation, even acceleration becomes a relative concept, at least on a small scale. This is made possible by the fact that, to an observer in a confined region of space, the effects of being accelerated and of being in a gravitational field are indistinguishable.

If two unaccelerated observers perform the same experiment, they must arrive at the same results. It makes no difference whether it is performed on the ground or in a smoothly travelling vehicle. However, if an accelerated observer performs the experiment, he may well get a different answer. The relativity principle asserts that all unaccelerated observers are equivalent: it says nothing about accelerated observers.

§1.1.C *Inertial Frames.* It is useful at this point to introduce the concept of a frame of reference. To specify positions and times, each observer may choose a zero of the time scale, an origin in space, and a set of three Cartesian co-ordinate axes. We shall refer to these collectively as a *frame of reference*. The position and time of any event may then be specified with respect to this frame by the three Cartesian co-ordinates x, y, z, and the time t. We may suppose, for example, that the observer is located on a solid body, that he chooses some point of this body as his origin, and takes his axes to be rigidly fixed to it.

In view of the relativity principle, the frames of reference used by different unaccelerated observers are completely equivalent. The laws of physics expressed in terms of x, y, z, t must be identical with those in terms of the co-ordinates of another frame, x', y', z', t'. They are not, however, identical with the laws expressed in terms of the co-ordinates used by an accelerated observer. The frames used by unaccelerated observers are called *inertial* frames.

We have not yet said how we can tell whether a given observer is unaccelerated. We need a criterion to distinguish inertial frames from the others. Formally, an inertial frame may be defined to be one with respect to which an isolated body, far removed from all other matter, would move with uniform velocity. This is of course an idealized definition, since in practice we never can get infinitely far away from other matter. For all practical purposes, however, an inertial frame is one whose orientation is fixed relative to the 'fixed' stars, and in which the sun (or, more precisely, the centre of mass of the solar system) moves with uniform velocity. It is an essential assumption of classical mechanics that such frames exist. Indeed, this assumption (together with a definition of inertial frames) is the real physical content of Newton's first law (a body acted on by no forces moves with uniform velocity in a straight line).

It is generally convenient to use only inertial frames, but there is no necessity to do so. Sometimes it proves convenient to use a non-inertial (in particular, rotating) frame, in which the laws of mechanics take on a more complicated form. For example, we shall discuss in Chapter 5 the use of a frame rigidly fixed to the earth. ? 73

§1.1.D *Vectors.* It is often convenient to use a notation which does not

refer explicitly to a particular set of co-ordinate axes. Instead of using Cartesian co-ordinates x, y, z, we may specify the position of a point P with respect to a given origin O by the length and direction of the line OP. A quantity which is specified by a magnitude and a direction is called a *vector;* in this case the *position vector* $\mathbf{r}$ of P with respect to O. Many other physical quantities are also vectors: examples are velocity and force. They are to be distinguished from *scalars*—like mass and energy—which are completely specified by a magnitude alone.

We shall assume here that the reader is familiar with the ideas of vector algebra; if not, he will find a discussion which includes all the results we shall need in Appendix A. ?

Throughout this book, vectors will be denoted by boldface letters (like $\mathbf{a}$). The magnitude of the vector will be denoted by the corresponding letter in italic type (a), or by the use of vertical bars ($|\mathbf{a}|$). The scalar and vector products of two vectors $\mathbf{a}$ and $\mathbf{b}$ will be written $\mathbf{a}\cdot\mathbf{b}$ and $\mathbf{a}\wedge\mathbf{b}$ respectively. We shall use $\hat{\mathbf{r}}$ to denote the unit vector in the direction of $\mathbf{r}$, $\hat{\mathbf{r}} = \mathbf{r}/r$. The unit vectors along the x-, y-, z-axes will be denoted by $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, so that

$$\mathbf{r} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}.$$

We shall use the vector notation in formulating the basic laws of mechanics, both because of the mathematical simplicity thereby attained, and because the physical ideas behind the mathematical formalism are often much clearer in terms of vectors.

1.2 Newton's Laws

The basic laws discussed in this section describe the way in which the position $\mathbf{r}(t)$ of a body changes with the time t. They can be applied to bodies of any size or shape, but only when we have specified what we mean by the 'position' of such a body. Only in the case of point particles (which do not exist in nature) does this concept have an intuitively obvious meaning. We shall therefore consider at this stage only small bodies which can effectively be located at a point, and postpone to Chapter 8 the problem of the motion of extensive bodies. We shall find there that the laws themselves prescribe the interpretation of the 'position' of any body.

We shall begin by simply stating Newton's laws, and defer to the following section a discussion of the physical significance of the concepts involved, particularly those of *mass* and *force*.

Let us consider an isolated system comprising N bodies, which we label by an index $i = 1, 2, \ldots, N$. We shall denote the position of the

ith body with respect to a given inertial frame by $\mathbf{r}_i(t)$. Its velocity and acceleration are

$$\mathbf{v}_i(t) = \dot{\mathbf{r}}_i(t),$$

$$\mathbf{a}_i(t) = \dot{\mathbf{v}}_i(t) = \ddot{\mathbf{r}}_i(t),$$

where the dots denote differentiation with respect to the time t. For example

$$\dot{\mathbf{r}} \equiv \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}.$$

Each body is characterized by a scalar constant, its *mass* m_i. Its *momentum* $\mathbf{p}_i$ is defined to be

$$\mathbf{p}_i = m_i \mathbf{v}_i.$$

The equation of motion, which specifies how the body will move with time, is Newton's second law

$$\dot{\mathbf{p}}_i = m_i \mathbf{a}_i = \mathbf{F}_i, \tag{1.1}$$

where $\mathbf{F}_i$ is the total force acting on the body. This force is composed of a sum of forces due to each of the other bodies in the system. If we denote the force *on* the ith body *due to* the jth body by $\mathbf{F}_{ij}$, then

$$\mathbf{F}_i - \mathbf{F}_{i1} + \mathbf{F}_{i2} + \ldots + \mathbf{F}_{iN} = \sum_{j=1}^{N} \mathbf{F}_{ij}, \tag{1.2}$$

where, of course, $\mathbf{F}_{ii} = \mathbf{0}$, since there is no force on the ith body due to itself. Note that since the right side of (1.2) is a vector sum, this equation incorporates the 'parallelogram law' of composition of forces.

The two-body forces $\mathbf{F}_{ij}$ must satisfy Newton's third law, which asserts that 'action' and 'reaction' are equal and opposite,

$$\mathbf{F}_{ji} = -\mathbf{F}_{ij}. \tag{1.3}$$

Moreover, $\mathbf{F}_{ij}$ is a function of the positions and velocities (and internal structure) of the ith and jth bodies, but is unaffected by the presence of other bodies.* Because of the relativity principle, it can in fact depend only on the *relative* position

$$\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j,$$

* It can be argued that this is an unnecessarily restrictive assumption. It would be perfectly possible to include also, say, three-body forces, which depend on the positions and velocities of three bodies simultaneously. However, within the realm of validity of classical mechanics, no such forces are known, and their inclusion would only serve as an inessential complication. There is some evidence to suggest that the forces which bind together the particles in atomic nuclei are of this more complicated type, but this is a problem to which classical mechanics is quite inapplicable.

(see Fig. 1.1) and the *relative* velocity

$$\mathbf{v}_{ij} = \dot{\mathbf{r}}_{ij} = \dot{\mathbf{r}}_i - \dot{\mathbf{r}}_j.$$

All that remains is to specify the precise laws by which these two-body forces are to be determined. The most important class of forces are the *central, conservative* forces, which depend only on the relative position of the two bodies, and have the form

$$\mathbf{F}_{ij} = \hat{\mathbf{r}}_{ij} f(r_{ij}), \tag{1.4}$$

where, as usual, $\hat{\mathbf{r}}_{ij}$ is the unit vector in the direction of $\mathbf{r}_{ij}$, and $f(r_{ij})$ is a scalar function of the relative distance r_{ij}. If $f(r_{ij})$ is positive, the force $\mathbf{F}_{ij}$ is a *repulsive* force directed outwards along the line joining the bodies; if $f(r_{ij})$ is negative, it is an *attractive* force, directed inwards.

According to Newton's law of universal gravitation, there is a force of this type between *every* pair of bodies, proportional in magnitude to the product of their masses. It is given by (1.4) with

$$f(r_{ij}) = -\frac{Gm_i m_j}{r_{ij}^2}, \tag{1.5}$$

where G is the gravitational constant, whose value is

$$\begin{aligned} G &= 6{\cdot}67 \times 10^{-8} \text{ dyn cm}^2 \text{ g}^{-2} \\ &= 6{\cdot}67 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}. \end{aligned}$$

Since the masses are always positive, this force is always attractive.

In addition, if the bodies are electrically charged, there is an electrostatic force given by

$$f(r_{ij}) = \frac{\alpha q_i q_j}{r_{ij}^2}, \tag{1.6}$$

where q_i and q_j are their electric charges, and α is a constant, analogous to G, whose numerical value depends on the choice of units of charge. In this book we shall generally use *electrostatic* units, defined by the special choice $\alpha = 1$. Frequently, however, we shall also quote the results in terms of SI units, in which

$$\alpha = 1/4\pi\varepsilon_0 = 9{\cdot}00 \times 10^9 \text{ N m}^2 \text{ C}^{-2}.$$

Electric charges may be of either sign, and therefore the electrostatic force may be repulsive or attractive according to the relative sign of q_i and q_j. Note the enormous difference in the orders of magnitude of the numerical values of G and α when expressed in SI units. This serves to illustrate the fact that gravitational forces are really

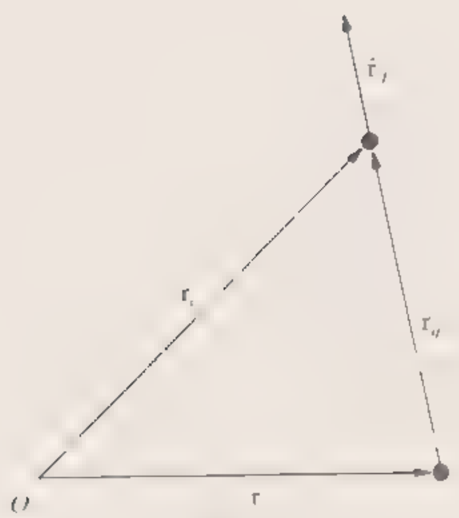


Fig. 1.1

exceptionally weak. They appear significant to us only because we happen to live close to a body of very large mass. Correspondingly large charges never appear, because positive and negative charges largely cancel out, leaving macroscopic bodies with a net charge close to zero.

In bodies with structure, central, conservative forces between their constituent parts can evidently give rise to forces which are still conservative (i.e. which are independent of velocity, and satisfy some further conditions that need not worry us here*), but no longer central (i.e. not directed along the line joining the bodies). This can happen, for example, if there is a distribution of electric charge within each body. They can also give rise, in a less obvious way, to non-conservative, velocity-dependent forces, as we shall see in Chapter 2.

Many resistive and frictional forces can be understood as macroscopic effects of forces which are really conservative on a small scale. The chief distinguishing feature of conservative forces is the existence of a conserved quantity, the energy of the system. Frictional forces have the effect of transferring some of this energy from the large-scale motion of the bodies to small-scale movements in their interior, and therefore appear non-conservative on a large scale.

In a sense, therefore, we may regard central, conservative forces as the norm. It would be wrong to conclude, however, that we can explain everything in terms of them. In the first place, the concepts of classical mechanics cannot be applied to the really small-scale structure of matter. For that, we need quantum mechanics. More serious is the existence of electromagnetic forces, which are of great importance even in the field of classical physics, but which cannot readily be accommodated in the framework of classical mechanics. The force between two charges in relative motion is neither central nor conservative, and does not even satisfy Newton's third law (1.3). This is a consequence of the finite velocity of propagation of electromagnetic waves. The force on one charge depends not on the instantaneous position of the other, but on its past history. The effect of a disturbance of one charge is not felt immediately by the other, but after an interval of time sufficient for a light signal to propagate from one to the other. This particular difficulty may be resolved by introducing the concept of the electromagnetic *field*. Then we may suppose that one charge does not act directly on the other, but on the field in its immediate vicinity; this in turn affects the field farther out, and so on. By supposing that the field itself can carry energy

* See Chapter 3 and Appendix A.

and momentum, we can reinstate the conservation laws, which are among the most important consequences of Newton's laws.

However, this does not completely remove the difficulty, for there is still an apparent contradiction between this classical electromagnetic theory and the principle of relativity discussed in §1.1. This arises from the fact that if the velocity of light is a constant with respect to one inertial frame—as it should be according to electromagnetic theory—then the usual rules for combining velocities would lead to the conclusion that it is not constant with respect to a relatively moving frame, in contradiction with the statement that all inertial frames are equivalent. This paradox can only be resolved by the introduction of Einstein's theory of relativity (i.e. 'special' relativity). Classical electromagnetic theory and classical mechanics *can* be incorporated into a single theory, but only by ignoring the relativity principle, and sticking to one inertial frame.

Reference from § 8·1 Momentum; Centre-of-Mass Motion.

1.3 The Concepts of Mass and Force

It is an important principle of physics that no quantity should be introduced into the theory which cannot, at least in principle, be measured. Now, Newton's laws involve not only the concepts of velocity and acceleration, which can be measured by measuring distances and times, but also the new concepts of mass and force. To give the laws a physical meaning we have, therefore, to show that these are measurable quantities. This is not quite as trivial as it might seem at first sight, for it is easy to see that any experiment designed to measure these quantities must necessarily involve Newton's laws themselves in its interpretation. Thus the operational definitions of mass and force—that is, the prescriptions of how they may be measured—which are required to make the laws physically significant, are actually contained in the laws themselves. This is by no means an unusual or logically objectionable situation, but it may clarify the status of these concepts to reformulate the laws in such a way as to isolate their definitional element.

Let us consider first the measurement of mass. Since the units of mass are arbitrary, we have to specify a way of comparing the masses of two given bodies. It is important to realize that we are discussing here the *inertial* mass, which appears in Newton's second law (1.1), and not the *gravitational* mass, which appears in (1.5). The two are of course proportional, but this is a physical law derived from experimental observation (in particular from Galileo's observation that all bodies fall equally fast) rather than an *a priori* assumption.

To verify the law, we must be able to measure each kind of mass separately. This rules out, for example, the use of a balance, which compares gravitational masses.

Clearly, we can compare the inertial masses of two bodies by subjecting them to equal forces and comparing their accelerations, but this does not help unless we have some way of knowing that the forces are equal. However, there is one case in which we *do* know this, because of Newton's third law. If we isolate the two bodies from all other matter, and compare their mutually induced accelerations, then according to (1.1) and (1.3),

$$m_1\mathbf{a}_1 = -m_2\mathbf{a}_2, \tag{1.7}$$

so that the accelerations are oppositely directed, and inversely proportional to the masses. This is in fact one of the commonly used methods for comparing masses, for example of elementary particles. If we allow two small bodies to collide, then during the collision the effects of more remote bodies are generally negligible in comparison with their effect on each other, and we may treat them approximately as an isolated system. (Such collisions will be discussed in detail in Chapters 2 and 7.) The mass ratio can then be determined from measurements of their velocities before and after the collision, by using (1.7) or its immediate consequence, the law of conservation of momentum,

$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = \text{constant}. \tag{1.8}$$

If we wish to separate the definition of mass from the physical content of equation (1.7), we may adopt as a fundamental law the following:

> In an isolated two-body system, the accelerations always satisfy the relation $\mathbf{a}_1 = -k_{21}\mathbf{a}_2$, where the scalar k_{21} is, for two given bodies, a constant independent of their positions, velocities and internal states.

If we choose the first body to be a standard body, and conventionally assign it unit mass (say 1 g), then we may *define* the mass of the second body to be $m_2 = k_{21}$ in units of this standard mass.

We must also assume of course that if we compare the masses of three bodies in this way, we obtain consistent results:

> For any three bodies, the constants k_{ij} satisfy the relation $k_{31} = k_{32}k_{21}$.

It then follows that for *any* two bodies, k_{32} is the mass ratio: $k_{32} = m_3/m_2$.

To complete the list of fundamental laws, we need a law which deals with systems containing more than two bodies, analogous to the law of composition of forces (1.2). This may be stated as follows:

> The acceleration induced in one body by another is some definite function of their positions, velocities and internal structure, and is unaffected by the presence of other bodies. In a many-body system, the acceleration of any given body is equal to the sum of the accelerations induced in it by each of the other bodies individually.

These laws, which appear in a rather unfamiliar form, are actually completely equivalent to Newton's laws, as stated in the previous section. In view of the apparently fundamental role played by the concept of force in Newtonian mechanics, it is remarkable that we have been able to reformulate the basic laws without mentioning this concept. It can of course be introduced, by defining it through Newton's second law (1.1). The utility of this concept arises from the fact that forces satisfy Newton's third law (1.3), while accelerations satisfy only the more complicated law (1.7). Since the mutually induced accelerations of two given bodies are always proportional, they are essentially determined by a single function, and it is useful to introduce the more symmetric concept of force, for which this becomes obvious.

It is interesting to note, finally, that one consequence of our basic laws is the additive nature of mass. Let us consider a three-body system. Then, returning to the notation of the previous section, the equations of motion for the three bodies are

$$m_1\mathbf{a}_1 = \mathbf{F}_{12}+\mathbf{F}_{13},$$
$$m_2\mathbf{a}_2 = \mathbf{F}_{21}+\mathbf{F}_{23},$$
$$m_3\mathbf{a}_3 = \mathbf{F}_{31}+\mathbf{F}_{32}.$$

If we add these equations, then, in view of (1.3), the terms on the right cancel in pairs, and we are left with

$$m_1\mathbf{a}_1+m_2\mathbf{a}_2+m_3\mathbf{a}_3 = \mathbf{0}, \tag{1.9}$$

which is the generalization of (1.7). Now, if we suppose that the force between the second and third is such that they are rigidly bound together to form a composite body, their accelerations must be equal: $\mathbf{a}_2 = \mathbf{a}_3$. In that case, we get

$$m_1\mathbf{a}_1 = -(m_2+m_3)\mathbf{a}_2,$$

which shows that the mass of the composite body is just m_2+m_3.

1.4 External Forces

To find the motion of the various bodies in any dynamical system, we have to solve two closely interrelated problems. First, given the positions and velocities at any one instant of time, we have to determine the forces acting on each body. Second, given the forces acting, we have to compute the new positions and velocities after a short interval of time has elapsed. In a general case, these two problems are inextricably bound up with each other, and must be solved simultaneously. If, however, we are concerned with the motion of a small body, or group of small bodies, then we can often neglect its effect on other bodies, and in that case the two problems can be separated.

For example, in discussing the motion of an artificial satellite, we can clearly ignore its effect on the earth. Since the motion of the earth is already known, we can calculate the force on the satellite as a function of its position and (if atmospheric resistance is included) its velocity. Then, taking the force as known, we can solve separately the problem of its motion. In this latter problem, we are really concerned with the satellite alone. The earth enters simply as a known external influence.

In many cases, therefore, it is useful to concentrate our attention on a small part of a dynamical system, and to represent the effect of everything outside this by external forces, which we suppose to be known in advance, as functions of position, velocity and time. This is the kind of problem with which we shall be mainly concerned in the next few chapters. Typically, we shall consider the motion of a particle under a known external force. In Chapter 6, we consider, for the gravitational and electrostatic cases, the complementary problem of determining the force from a knowledge of the positions of other bodies. Later, in Chapter 7, we return to the more complex type of problem in which the system of immediate interest cannot be taken to be merely a single particle.

1.5 Summary

To some extent the selection of a group of basic concepts, in terms of which others are to be defined, is a matter of choice. We have chosen to regard position and time (relative to some frame of reference) as basic. From this point of view, Newton's laws must be regarded as containing definitions in addition to physical laws. The first law contains the definition of an inertial frame, together with the physical assertion that such frames exist, while the second and

third contain the definitions of mass and force. These laws, supplemented by the laws of force, such as the law of universal gravitation, provide the equations from which we can determine the motion of any dynamical system.

PROBLEMS

1 The two components of a double star are observed to move in circles of radii a_1 and a_2. What is the ratio of their masses?

2 An object *A* moving with velocity $\mathbf{v}$ collides with a stationary object *B*. After the collision, *A* is moving with velocity $\frac{1}{2}\mathbf{v}$, and *B* with velocity $\frac{3}{2}\mathbf{v}$. Find the ratio of their masses. If, instead of bouncing apart, the two bodies stuck together after the collision, with what velocity would they then move?

3 Discuss the possibility of using force rather than mass as the basic quantity, taking for example a standard weight (at a given latitude) as the unit of force. How should one then define and measure the mass of a body?

Linear Motion

Chapter 2

In this chapter we discuss the motion of a body which is free to move only in one dimension. The problems considered are chosen to illustrate the concepts and techniques which will be of use in the more general case of three-dimensional motion.

2.1 Conservative Forces

We consider first a particle moving along a line, under a force which is given as a function of its position, $F(x)$. Then the equation of motion (1.1) is

$$m\ddot{x} = F(x). \tag{2.1}$$

Multiplying by $\dot{x}$, and integrating with respect to t, we obtain the equation

$$T+V = E = \text{constant}, \tag{2.2}$$

where T is the *kinetic energy*

$$T = \tfrac{1}{2}m\dot{x}^2 \tag{2.3}$$

and V is the *potential energy*

$$V(x) = -\int_{x_0}^{x} F(x)\,\mathrm{d}x. \tag{2.4}$$

Here x_0 is an arbitrary integration constant, and there is a corresponding arbitrary additive constant in V, and hence also in the *total energy* E. Equation (2.4) can be inverted to give the force in terms of the potential energy,

$$F(x) = -\frac{\mathrm{d}V}{\mathrm{d}x}. \tag{2.5}$$

$\Rightarrow \quad \mathbf{F}(\mathbf{r}) = -\nabla V?$

The equation (2.2) is the law of *conservation of energy*. A force of this type, depending only on x, is called a *conservative* force. A great deal of information about the motion can be obtained from this conservation law, even without integrating again to find x explicitly as a function of t. If the initial position and velocity are given, we can calculate the value of the constant E. Then (2.2) in the form

$$\tfrac{1}{2}m\dot{x}^2 = E - V(x) \tag{2.6}$$

gives the velocity of the particle (except for sign) when it is at any given position x. Since the kinetic energy is obviously positive, the motion is confined to the region where

$$V(x) \leqslant E.$$

For example, if $V(x)$ has the form shown in Fig. 2.1, and the value of E corresponds to the horizontal line, then the motion is confined either between x_1 and x_2, or between x_3 and $+\infty$. The velocity vanishes only at the points where $V(x) = E$, that is, where the curve crosses the line. Thus, if the particle starts from rest at x_1, it will move off to the right with increasing, and then decreasing, velocity until it reaches x_2, where it comes instantaneously to rest, and reverses direction. The motion is in this case a repeated oscillation between x_1 and x_2. If it starts from rest at x_3, it will move off

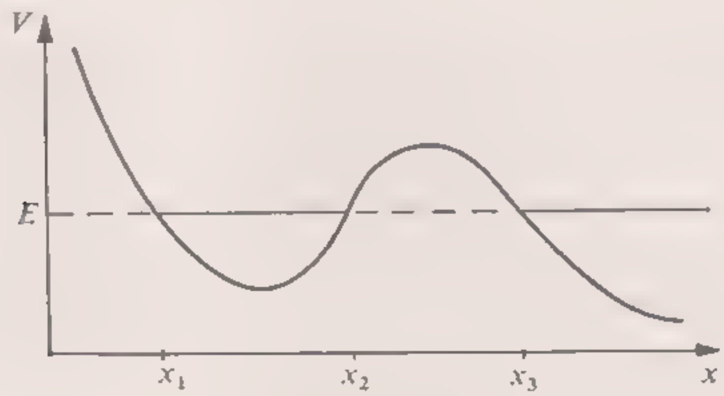


Fig. 2.1

with increasing velocity towards $+\infty$. The motion may conveniently be pictured as that of a particle sliding under gravity on the (smooth) potential energy curve.

To illustrate these ideas, we consider two special shapes of the potential energy curve. The first is the *potential well* illustrated in Fig. 2.2, which corresponds to an *attractive* force, directed towards the centre of the well. (Whenever possible it is convenient to choose the arbitrary constant in V so that $V(x) \to 0$ as $x \to \pm\infty$; we have done this here.) Two types of motion are possible. If the initial conditions are such that E is negative ($E_{(1)}$ in figure), then the motion is confined to a finite region, and the particle oscillates repeatedly between the two limiting points where $V(x) = E$. If, on the other hand, the particle starts from a point far out to the left with velocity v, then $E \approx \frac{1}{2}mv^2$ is positive ($E_{(2)}$ in figure), and the motion is not confined. The particle accelerates as it goes into the well, and decelerates as it emerges, finally reaching a point far to the right with velo-

city v equal to its initial velocity. Note that it will arrive somewhat earlier than it would have done had the well been absent.

The second special case is the inverted well, or *potential barrier*, illustrated in Fig. 2.3. This corresponds to a *repulsive* force, directed outwards from the centre. Again, two types of motion are possible.

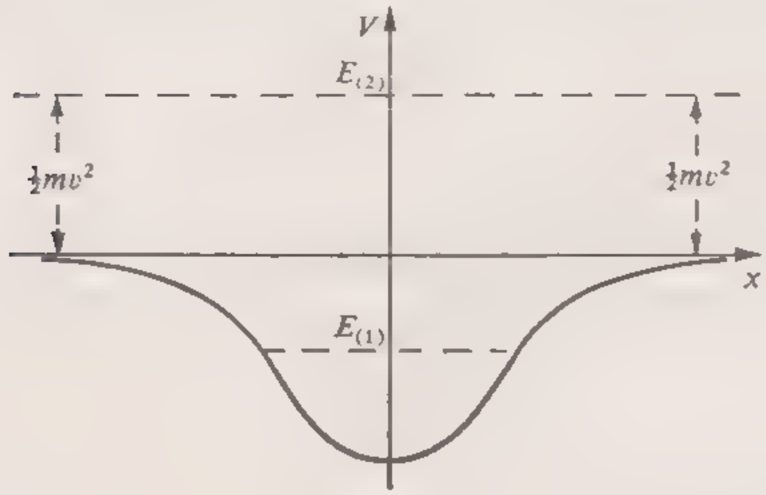


Fig. 2.2

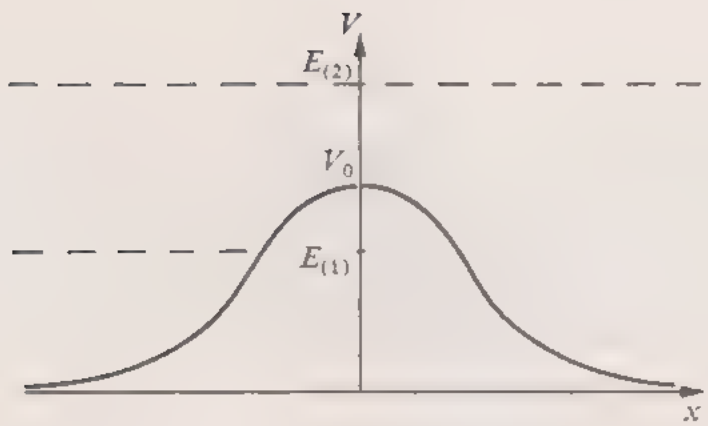


Fig. 2.3

If the particle starts far out to the left with velocity v such that $E \approx \frac{1}{2}mv^2 < V_0$, it will reach some finite point, reverse direction, and move off towards $-\infty$, finally emerging with velocity $-v$. On the other hand, if $\frac{1}{2}mv^2 > V_0$, the particle has sufficient energy to get over the barrier. It will arrive at a point far to the right with velocity v, but this time rather later than it would have done in the absence of the barrier.

2.2 Motion near Equilibrium; the Harmonic Oscillator

A particle can be in equilibrium only if the force acting on it is zero. For a conservative force, this means, by (2.5), that the potential energy curve is horizontal at the position of the particle. Let us now consider the motion of a particle near a position of equilibrium. Without loss of generality, we may choose the equilibrium point to be the origin $x = 0$, and choose the arbitrary constant in V so that $V(0) = 0$. If we are interested only in small displacements, we may expand $V(x)$ in a Maclaurin series,

$$V(x) = V(0)+xV'(0)+\tfrac{1}{2}x^2V''(0)+\ldots,$$

where the primes denote derivatives with respect to x. Since we have chosen $V(0) = 0$, the constant term is absent, and since the equilibrium condition is $V'(0) = 0$, the linear term is absent also. Thus near $x = 0$ we can write, approximately,

$$V(x) = \tfrac{1}{2}kx^2, \qquad k = V''(0). \tag{2.7}$$

(We assume for simplicity that $V''(0)$ does not also vanish.) By (2.5), the corresponding force is

$$F(x) = -kx,$$

so that the equation of motion is

$$m\ddot{x}+kx = 0. \tag{2.8}$$

This equation is very easy to solve. However, since we shall encounter a number of similar equations later, it will be useful to discuss the method of solution in some detail. It is a linear differential equation; that is, one involving only linear terms in x and its derivatives. Such equations have the important property that their solutions satisfy the *superposition principle*: if $x_1(t)$ and $x_2(t)$ are solutions, then so is any linear combination

$$x(t) = a_1x_1(t)+a_2x_2(t), \tag{2.9}$$

where a_1 and a_2 are constants; for, clearly,

$$m\ddot{x}+kx = a_1(m\ddot{x}_1+kx_1)+a_2(m\ddot{x}_2+kx_2) = 0.$$

Moreover, if x_1 and x_2 are linearly independent solutions (that is, unless x_2 is simply a constant multiple of x_1), then (2.9) is actually the general solution. For, since (2.8) is of second order, we could obtain its solution by integrating twice, and the general solution must therefore contain just two arbitrary constants of integration. All we have

to do, therefore, is to find any two independent solutions $x_1(t)$ and $x_2(t)$.

Let us consider first the case where $k<0$, so that $V(x)$ has a maximum at $x = 0$. Then (2.8) can be written

$$\ddot{x}-p^2x = 0, \qquad p = (-k/m)^{1/2}. \tag{2.10}$$

It is easy to verify that this equation is satisfied by the functions $x = e^{pt}$ and $x = e^{-pt}$. Thus the general solution is

$$x = \tfrac{1}{2}Ae^{pt}+\tfrac{1}{2}Be^{-pt}. \tag{2.11}$$

(We introduce the factor of $\frac{1}{2}$ for later convenience; it is a matter of convention whether we call the arbitrary constants A, B or $\frac{1}{2}A$, $\frac{1}{2}B$.) Clearly, a small displacement will in general lead to an exponential increase of x with time, which continues until the approximation involved in (2.7) ceases to be valid. Thus the equilibrium is unstable, as we should expect when V has a maximum.

We now turn to the case where $k>0$, and $V(x)$ has a minimum at $x = 0$. Then the potential energy function (2.7) is that of a *simple harmonic oscillator*. The equation of motion (2.8) becomes

$$\ddot{x}+\omega^2x = 0, \qquad \omega = (k/m)^{1/2}. \tag{2.12}$$

The functions $x = \cos \omega t$ and $x = \sin \omega t$ are obvious solutions of this equation, and the general solution is therefore

$$x = c \cos \omega t+d \sin \omega t. \tag{2.13}$$

The arbitrary constants c and d are to be determined by the initial conditions. If at $t = 0$ the particle is at x_0, with velocity v_0, then we easily find

$$c = x_0, \qquad d = v_0/\omega.$$

An alternative form of (2.13) is

$$x = a \cos (\omega t-\theta), \tag{2.14}$$

where the constants a, θ are related to c, d by

$$c = a \cos \theta, \qquad d = a \sin \theta.$$

The constant a is called the *amplitude*. It defines the extreme limits between which the particle oscillates, $x = \pm a$. The motion is a periodic oscillation, of *period* τ given by

$$\tau = \frac{2\pi}{\omega}. \tag{2.15}$$

(See Fig. 2.4.) The *frequency* ν is the number of oscillations per unit time, namely,

$$\nu = \frac{1}{\tau} = \frac{\omega}{2\pi}. \tag{2.16}$$

Notice that this discussion applies to the motion of a particle near an equilibrium point of *any* potential energy function. For sufficiently small displacements, any system of this kind behaves like a simple harmonic oscillator. In particular, the frequency or period of small oscillations may always be found from the second derivative of the potential energy function at the equilibrium position.

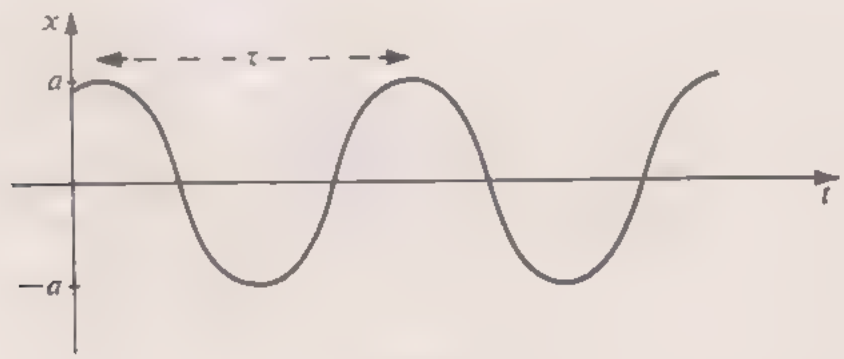


Fig. 2.4

2.3 Complex Representation

It is often very convenient, in discussing periodic phenomena, to use complex numbers. In particular, this allows us to treat both cases, $k>0$ and $k<0$, of the previous section together. If we allow p to be complex, then the solution in both cases is given by (2.10) and (2.11). When $k>0$, p becomes purely imaginary,

$$p = i\omega,$$

and the solution (2.11) is

$$x = \tfrac{1}{2}Ae^{i\omega t} + \tfrac{1}{2}Be^{-i\omega t}. \tag{2.17}$$

Of course, x must be a real number, so that A and B must be complex conjugates. If we write

$$A = c - id, \qquad B = c + id,$$

and use the relation

$$e^{i\omega t} = \cos \omega t + i \sin \omega t,$$

we recover the solution (2.13). Similarly, if we use the polar form of a complex number,

$$A = ae^{-i\theta}, \qquad B = ae^{i\theta},$$

we obtain the solution in the form (2.14).

Another useful expression can be obtained by noting that the sum of a number and its complex conjugate is equal to twice its real part; for example,

$$A+B = 2\,\mathrm{Re}(A) = 2c.$$

Since the two terms in (2.17) are complex conjugates, it can also be written

$$x = \mathrm{Re}(Ae^{i\omega t}). \tag{2.18}$$

We could also have obtained this solution directly by noting that if $z = x+iy$ is a complex solution of (2.8),

$$m\ddot{z}+kz = (m\ddot{x}+kx)+i(m\ddot{y}+ky) = 0,$$

then its real and imaginary parts, $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$, must separately satisfy the equation. This method of obtaining solutions will be useful later.

If we plot the complex solution

$$z = x+iy = Ae^{i\omega t}$$

in the xy-plane, we see that z moves in a circle around the origin with angular velocity ω. For this reason, ω is usually called the *angular frequency*. The constant A is a complex amplitude, whose absolute value (the real amplitude a) is the radius of the circle, and whose phase θ defines the initial direction of the vector from the origin to z. The real part of the solution, x, may be regarded as the projection of this circular motion on the real axis.

2.4 The Law of Conservation of Energy

This law was originally derived from Newton's laws for the case where the force is a function only of x. It has, however, a much wider application. By introducing additional forms of energy (heat, chemical, electromagnetic and so on), it has been extended far beyond the field of mechanics, to the point where it is now recognized as one of the most fundamental of all physical laws. The existence of such a law, and of the laws of conservation of momentum and angular momentum, is in fact closely related to the relativity principle, as we shall see later in Chapter 13.

Many non-conservative (or *dissipative*) forces may be regarded as macroscopic effects of forces which are really conservative on a small scale. For example, when a particle penetrates a retarding medium, such as the atmosphere, it experiences a force which is velocity-dependent, and therefore non-conservative. However, if we look at the situation on a sub-microscopic scale, we see that what happens is that the particle makes a series of collisions with the molecules of the medium. In each collision, energy is conserved, and some of the kinetic energy of the incoming particle is transferred to the molecule with which it collides. (Such collisions will be discussed in detail in Chapter 7.) By means of further collisions, this energy is gradually distributed among the surrounding molecules. The net result is to retard the incoming particle, and to increase the average energy of the molecules in the medium. This increased energy appears macroscopically as heat, and results in a rise in temperature of the medium.

For an arbitrary force F, the rate of change of the kinetic energy T is

$$\dot{T} = m\dot{x}\ddot{x} = F\dot{x}. \tag{2.19}$$

Thus the increase in kinetic energy in a time interval $\mathrm{d}t$, during which the particle moves a distance $\mathrm{d}x$, is

$$\mathrm{d}T = \mathrm{d}W, \tag{2.20}$$

where

$$\mathrm{d}W = F\,\mathrm{d}x \tag{2.21}$$

is called the *work* done by the force F in the infinitesimal displacement $\mathrm{d}x$. (Thus for a conservative force the potential energy $V(x)$ is equal to minus the work done by the force in the displacement from the fixed point x_0 to x.) The work done is therefore a measure of the amount of energy converted to kinetic energy from other forms. In a real mechanical system, there is usually some loss of mechanical (kinetic or potential) energy to heat or other forms. Correspondingly, there will be dissipative forces acting on the system.

2.5 The Damped Oscillator

We saw in §2.2 that a particle near a position of stable equilibrium under a conservative force may always be treated approximately as a simple harmonic oscillator. If there is energy loss, we must include in the equation of motion a force depending on the velocity. So long as we are concerned only with small displacements from the equilibrium position, we may treat both x and $\dot{x}$ as small quantities, and neglect

x^2, $x\dot{x}$ and $\dot{x}^2$ in comparison with x and $\dot{x}$. Thus we are led to consider the *damped harmonic oscillator* for which

$$F = -kx - \alpha\dot{x},$$

where α is another constant. Thus the equation of motion becomes

$$m\ddot{x} + \alpha\dot{x} + kx = 0. \tag{2.22}$$

Equations of this form turn up in many branches of physics. For example, the oscillations of an electrical circuit containing an inductance L, resistance R and capacitance C in series are described by the equation

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = 0,$$

in which the variable q represents the charge on one plate of the condenser.

The equation (2.22) may be solved as before by looking for solutions of the form

$$x = e^{pt}.$$

Substituting in (2.22), we obtain for p the equation

$$mp^2 + \alpha p + k = 0.$$

The solutions of this equation are

$$p = -\gamma \pm (\gamma^2 - \omega_0{}^2)^{1/2}, \tag{2.23}$$

where

$$\gamma = \alpha/2m \tag{2.24}$$

and ω_0 is the angular frequency of the undamped oscillator,

$$\omega_0 = (k/m)^{1/2}. \tag{2.25}$$

The rate at which work is done by the force $-\alpha\dot{x}$ is $-\alpha\dot{x}^2$. If α were negative, the particle would, therefore, be gaining, rather than losing, energy. So we shall assume that α is positive.

Large Damping. If α is so large that $\gamma^2 > \omega_0{}^2$, then both roots for p are real and negative,

$$p = -\gamma_\pm, \qquad \gamma_\pm = \gamma \pm (\gamma^2 - \omega_0{}^2)^{1/2}.$$

The general solution is then

$$x = \tfrac{1}{2}Ae^{-\gamma_+ t} + \tfrac{1}{2}Be^{-\gamma_- t}, \tag{2.26}$$

where A and B are arbitrary constants. Hence the displacement tends exponentially to zero. For large times, the dominant term is that containing in the exponent the smaller quantity γ_-. Thus the characteristic time in which x is reduced by a factor $1/e$ is of the order of $1/\gamma_-$.

Small Damping. Now let us consider the case where α is small, so that $\gamma^2 < \omega_0^2$. Then the two roots for p are complex conjugates,

$$p = -\gamma \pm i\omega, \qquad \omega = (\omega_0^2 - \gamma^2)^{1/2}. \tag{2.27}$$

The general solution may be written in the alternative forms

$$\begin{aligned} x &= \tfrac{1}{2}Ae^{i\omega t - \gamma t} + \tfrac{1}{2}Be^{-i\omega t - \gamma t} \\ &= \mathrm{Re}(Ae^{i\omega t - \gamma t}) \\ &= ae^{-\gamma t} \cos(\omega t - \theta), \end{aligned} \tag{2.28}$$

where

$$A = ae^{-i\theta}, \qquad B = ae^{i\theta}.$$

From the last form of (2.28), we see that this solution represents an oscillation with exponentially decreasing amplitude $ae^{-\gamma t}$, and angular frequency ω. (See Fig. 2.5.) Note that ω is less than the frequency ω_0 of the undamped oscillator. The time in which the amplitude is reduced by a factor $1/e$ is

$$1/\gamma = 2m/\alpha.$$

This is called the *relaxation time* of the oscillator.

It is often convenient to introduce the *quality factor*, or simply 'Q', of the resonance, defined as the dimensionless number

$$Q = m\omega_0/\alpha = \omega_0/2\gamma.$$

If the damping is small, then Q is large. In a single oscillation period, the amplitude is reduced by the factor $e^{-2\pi\gamma/\omega}$, or approximately $e^{-\pi/Q}$. The number of periods in a relaxation time is Q/π.

Critical Damping. The limiting case, $\gamma^2 = \omega_0^2$, is the case of *critical* damping, in which $\omega = 0$ and the two roots for p coincide. Then the solution (2.26) or (2.28) involves only one arbitrary constant, $A+B$, and cannot be the general solution. We need to find a second independent solution in this case. In fact, it is easy to verify by direct

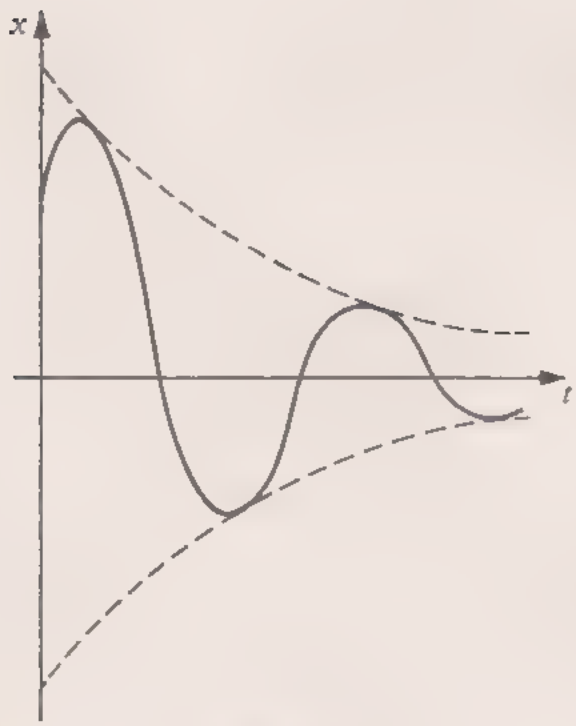


Fig. 2.5

substitution that the equation (2.22) is also satisfied by the function $te^{-\gamma t}$. Thus the general solution is

$$x = (a+bt)e^{-\gamma t}. \tag{2.29}$$

Critical damping is often the ideal. For example, in a measuring instrument we want to damp out the oscillations of the pointer about its correct position as quickly as possible, but too much damping would lead to a very slow response. Let us assume that k is fixed, and the amount of damping varied. When the damping is less than critical ($\gamma < \omega_0$), the characteristic time of response is the relaxation time $1/\gamma$, which of course *decreases* as γ is increased. However, when $\gamma > \omega_0$, the characteristic time is $1/\gamma_-$, as we noted above. It is not hard to verify that as γ increases, γ_- decreases, so that the response time $1/\gamma_-$ *increases*. Thus the shortest possible response time is obtained by choosing $\gamma = \omega_0$, that is for critical damping.

2.6 Oscillator under Periodic Force

In an isolated system, the forces are functions of position and velocity, but not explicitly of the time. However, we are often interested in the response of an oscillatory system to an applied external force, which is given as a function of the time. Then we have to consider the equation

$$m\ddot{x}+\alpha\dot{x}+kx = F(t), \tag{2.30}$$

where $F(t)$ is the external force. Now, if $x_1(t)$ is any solution of this equation, and $x_0(t)$ is a solution of the corresponding homogeneous equation (2.22) for the unforced oscillator, then $x_1(t)+x_0(t)$ will be another solution of (2.30). Hence we only need to find *one* particular solution of this equation. The general solution is then obtained by adding to this particular solution the general solution of the homogeneous equation (2.22). (This will be the *general* solution because it contains *two* arbitrary constants.)

Simple Periodic Force—Resonance. Let us consider first the case where the applied force is periodic in time, with the simple form

$$F(t) = F_1 \cos \omega_1 t, \tag{2.31}$$

where F_1 and ω_1 are constants. It is convenient to write this in the form

$$F(t) = \mathrm{Re}(F_1\, e^{i\omega_1 t}),$$

and to solve first the equation with a complex force

$$m\ddot{z}+\alpha\dot{z}+kz = F_1\,e^{i\omega_1 t}. \tag{2.32}$$

Then the real part x of the this solution will be a solution of the equation (2.30) with the force (2.31).

We now look for a solution of (2.32) which is periodic in time, with the same period as the applied force,

$$z = A_1\,e^{i\omega_1 t} = a_1\,e^{i(\omega_1 t-\theta_1)},$$

where $A_1 = a_1 e^{-i\theta_1}$ is a complex constant. Substituting in (2.32) we obtain

$$(-m\omega_1{}^2+i\alpha\omega_1+k)A_1 = F_1, \tag{2.33}$$

or, dividing by $me^{-i\theta_1}$, and rearranging the terms,

$$(\omega_0{}^2+2i\gamma\omega_1-\omega_1{}^2)a_1 = \frac{F_1}{m}\,e^{i\theta_1},$$

where γ and ω_0 are defined as before by (2.24) and (2.25). Hence, equating real and imaginary parts,

$$(\omega_0{}^2-\omega_1{}^2)a_1 = \frac{F_1}{m}\cos\theta_1,$$

$$2\gamma\omega_1 a_1 = \frac{F_1}{m}\sin\theta_1.$$

Thus the amplitude a_1 and phase θ_1 of our solution are given by

$$a_1 = \frac{F_1/m}{[(\omega_0{}^2-\omega_1{}^2)^2+4\gamma^2\omega_1{}^2]^{1/2}}, \tag{2.34}$$

$$\tan\theta_1 = \frac{2\gamma\omega_1}{\omega_0{}^2-\omega_1{}^2}. \tag{2.35}$$

We have found a particular solution of (2.32). Its real part, that is, the corresponding particular solution of our original equation, is simply

$$x = a_1\cos(\omega_1 t-\theta_1). \tag{2.36}$$

The general solution is obtained by adding to this particular solution the general solution of the corresponding homogeneous equation,

namely (2.26) or (2.28). In the case where the damping is less than critical ($\gamma^2 < \omega_0{}^2$), we obtain

$$x = a_1 \cos(\omega_1 t - \theta_1) + a e^{-\gamma t} \cos(\omega t - \theta). \qquad (2.37)$$

Here, of course, a and θ are arbitrary constants, while a_1 and θ_1 are fixed by (2.34) and (2.35).

The second term in the general solution (2.37), which represents a free oscillation, dies away exponentially with time. It is therefore called the *transient*. After a long time, the displacement x will be given by the first term of (2.37), that is by (2.36), alone. Thus, no matter what initial conditions we choose, the oscillations are ultimately governed solely by the external force. Note that their period is the period of the applied force, not the period of the unforced oscillator.

The amplitude a_1 and phase θ_1 of the forced oscillations are strongly dependent on the angular frequencies ω_0 and ω_1. In particular, if the damping γ is small, the amplitude can become very large when the frequencies are almost equal. If we fix γ and the forcing frequency ω_1, and vary the oscillator frequency ω_0, the amplitude is a maximum when $\omega_0 = \omega_1$. In this case, we say that the system is in *resonance*. At resonance, the amplitude is

$$a_1 = \frac{F_1}{2m\gamma\omega_1} = \frac{F_1}{\alpha\omega_1}, \qquad (2.38)$$

which can be very large if the damping constant α is small. If we fix the parameters γ and ω_0 of the oscillator, and vary the forcing frequency ω_1, the maximum amplitude actually occurs for a frequency ω_1 slightly lower than ω_0, namely

$$\omega_1{}^2 = \omega_0{}^2 - 2\gamma^2.$$

However, if γ is small, this does not differ much from ω_0. Note that the natural frequency ω of the oscillator lies between this resonant frequency and the natural frequency ω_0 of the undamped oscillator.

Near resonance, the dependence of the amplitude a_1 on the forcing frequency ω_1 is of the form illustrated in Fig. 2.6. The *width* of the resonance, that is the range of frequencies over which the amplitude is large, is determined by γ. For, the amplitude is reduced to $1/\sqrt{2}$ of its peak value when the two terms in the denominator of (2.34) become comparable in magnitude, and for small γ this occurs when $\omega_1 = \omega_0 \pm \gamma$. Therefore, γ is called the *half-width* of the reson-

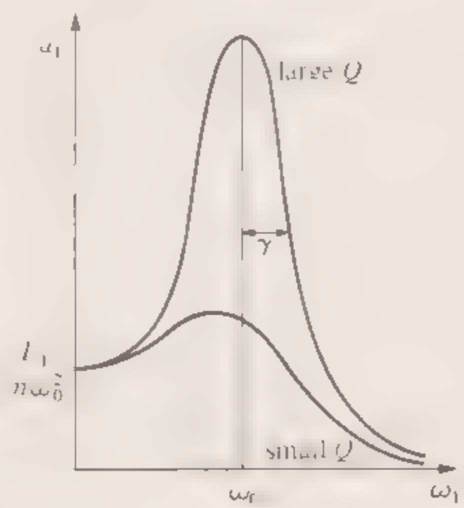


Fig. 2.6

ance. Notice the inverse relationship between width and peak amplitude: the narrower the resonance, the higher is its peak.

The quality factor $Q = \omega_0/2\gamma$ provides a quantitative measure of the sharpness of the resonance peak. Indeed, the ratio of the amplitude at resonance, given by (2.38), to the amplitude at $\omega_1 = 0$ (which describes the response to a time-independent force) is precisely Q.

This phenomenon of resonance occurs with any oscillatory system, and is of great practical importance. Since quite small forces can set up large oscillations if the frequencies are in resonance, great care must be taken in the design of any mechanical structure to avoid this possibility. It would be undesirable, for example, to build a ship whose natural frequency of pitching coincided with the frequency of the waves it is likely to encounter.

The constant θ_1 specifies the phase relation between the applied force and the induced oscillations. If the force is slowly oscillating, ω_1 is small, and $\theta_1 \approx 0$, so that the induced oscillations are in phase with the force. In this case, the amplitude (2.34) is

$$a_1 \approx \frac{F_1}{m\omega_0^2} = \frac{F_1}{k}.$$

$F_1 \cos(\omega_1 t - kx)$

Thus the position x at any time t, given by (2.36), is approximately the equilibrium position under the force $F_1 \cos\omega_1 t - kx$. At resonance, the phase shift is $\theta_1 = \frac{1}{2}\pi$, and the induced oscillations lag behind by a quarter period. (The variation of θ_1 with ω_1 is shown in Fig. 2.7.) For very rapidly oscillating forces, $\theta_1 \approx \pi$, and the oscillations are almost exactly out of phase. In this limiting case $a_1 \approx F_1/m\omega_1^2$, and the oscillations correspond to those of a free particle under the applied oscillatory force. Note that the value of the damping term γ is important only in the region near the resonance.

General Periodic Force. The solution we have obtained can immediately be generalized to the case where the applied force is a sum of periodic terms,

$$F(t) = \sum_r F_r e^{i\omega_r t}. \tag{2.39}$$

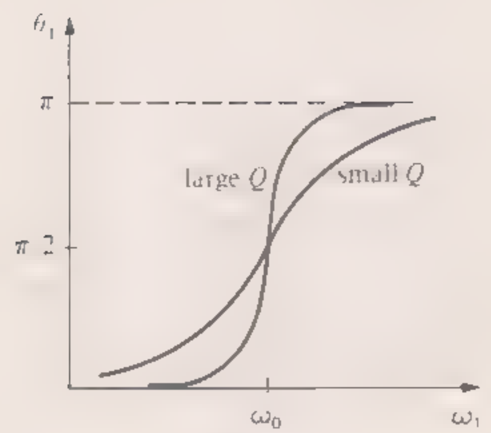


Fig. 2.7

We can always ensure that this force is real by including with each term its complex conjugate, so that it is not really necessary to write Re in front of the sum. (In particular a simple periodic force may be written as a sum of two terms proportional to $e^{i\omega_1 t}$ and $e^{-i\omega_1 t}$.) Because of the linearity of (2.30), the corresponding solution is

$$x = \sum_r A_r e^{i\omega_r t} + \text{transient}, \tag{2.40}$$

where each A_r is related to the corresponding F_r by (2.33). This may easily be verified by direct substitution in (2.30).

In particular, if we take

$$F(t) = \sum_{n} F_n e^{in\omega t}, \tag{2.41}$$

from page 192 and read to (2-42) from 'General Periodic Force'.

where the sum is over all integers n, then we have a periodic force,

$$F(t+\tau) = F(t), \qquad \tau = 2\pi/\omega. \tag{2.42}$$

Actually, *any* periodic function with period τ can be expanded in the form (2.41), which is called a *Fourier series.* If we are given $F(t)$, it is easy to find the corresponding coefficients F_n. For one easily sees that

$$\frac{1}{\tau}\int_0^\tau e^{i(n-m)\omega t}\,dt = \delta_{nm}, \tag{2.43}$$

where δ_{nm} is Kronecker's symbol defined by

$$\delta_{nm} = \begin{cases} 1 & \text{if} \quad m = n, \\ 0 & \text{if} \quad m \neq n. \end{cases} \tag{2.44}$$

Thus if we multiply (2.41) by $e^{-im\omega t}$ and integrate over a period, the only term in the sum which survives is that with $n = m$. Hence we obtain

$$F_n = \frac{1}{\tau}\int_0^\tau F(t)e^{-in\omega t}\,dt. \tag{2.45}$$

For example, suppose that the force has the 'square-wave' form shown in Fig. 2.8, and given by

$$F(t) = \begin{cases} F, & n\tau < t < (n+\frac{1}{2})\tau, \\ -F, & (n-\frac{1}{2})\tau < t < n\tau, \end{cases}$$

for $n = 0, \pm 1, \pm 2, \ldots$. Then, since $\omega = 2\pi/\tau$, we find

$$\begin{aligned} F_n &= \frac{F}{\tau}\int_0^{\frac{1}{2}\tau} e^{-2\pi int/\tau}\,dt - \frac{F}{\tau}\int_{\frac{1}{2}\tau}^{\tau} e^{-2\pi int/\tau}\,dt \\ &= \frac{F}{i\pi n}[1-(-1)^n]. \end{aligned}$$

Note that $F_n = 0$ for all even values of n. This happens because in addition to repeating after the period τ, $F(t)$ repeats with opposite sign after half this period, $F(t+\frac{1}{2}\tau) = -F(t)$. For odd values of n, we find from (2.33) that the amplitudes A_n are

$$A_n = \frac{2F}{i\pi nm(\omega_0{}^2 - n^2\omega^2 + 2in\gamma\omega)}, \qquad (n \text{ odd}).$$

The position of the oscillator is then given by

$$x = \sum_{r=-\infty}^{+\infty} A_{2r+1}\, e^{i(2r+1)\omega t} + \text{transient}.$$

If ω_0 is close to one of the values $n\omega$, then one pair of amplitudes A_n and A_{-n} will be much larger than the rest, and the oscillation

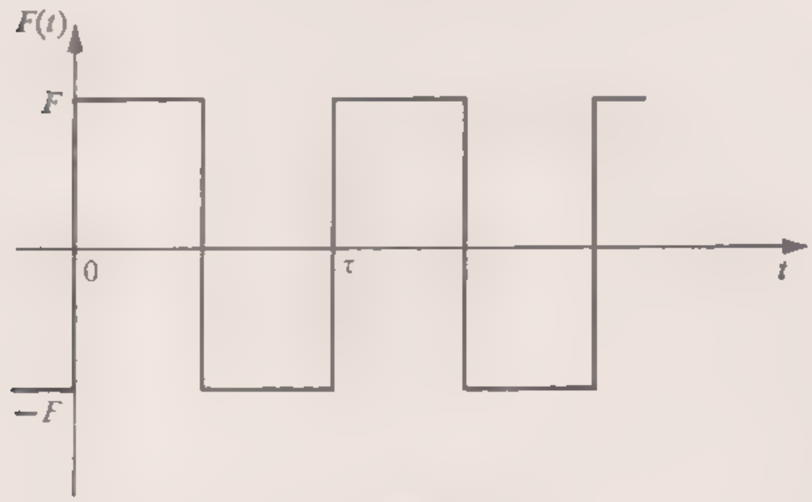


Fig. 2.8

will be almost simple harmonic motion at this frequency. In general, the amplitudes A_n decrease rapidly in magnitude with increasing values of n, so that we can obtain a good approximation to x by keeping only a few terms of the series.

In this way, one can obtain the solution to (2.30) for an arbitrary periodic force. As a matter of fact, this kind of treatment can be extended even further, by replacing the sum in (2.39) by an integral over all possible angular frequencies ω. Any force whatever, subject to some very general mathematical requirements, can be written in this form. (This is the Fourier integral theorem.) However, we shall not pursue this method further here, because we shall find in the next section an alternative method of solving the problem, which is often simpler to use.

2.7 Impulsive Forces; the Green's Function Method

There are many physical situations, for example collisions, in which very large forces act for very short times. Let us consider a force F acting on a particle during the time interval Δt. The resulting change in momentum of the particle is

$$\Delta p = p(t+\Delta t)-p(t) = \int_{t}^{t+\Delta t} F\,\mathrm{d}t. \tag{2.46}$$

The quantity on the right is called the *impulse* I delivered to the particle. It is natural to consider an idealized situation in which the time interval Δt tends to zero, and the whole of the impulse I is delivered to the particle instantaneously. In other words, we let $\Delta t \to 0$ and $F \to \infty$ in such a way that I remains finite. At the instant when the impulse is delivered, the momentum of the particle changes discontinuously. Although in reality the force must always remain finite, this is a good description of, for example, the effect of a sudden blow.

In the following section we shall discuss some simple collision problems. Here, we shall consider the effect of an impulsive force on an oscillator. Let us suppose first that the oscillator is at rest at its equilibrium position $x = 0$, and that at time $t = 0$ it experiences a blow of impulse I. Immediately after the blow, its position is still $x = 0$ (since the velocity remains finite, the position does not change discontinuously), while its velocity is $v_0 = I/m$. Inserting these initial conditions in the general solution (2.28) for the oscillatory case $\gamma^2 < \omega_0{}^2$, we find that the position is given by

$$\begin{aligned} x &= 0, && t<0; \\ x &= \frac{I}{m\omega}\,\mathrm{e}^{-\gamma t}\sin \omega t, && t>0. \end{aligned} \tag{2.47}$$

This solution is illustrated in Fig. 2.9.

Just as in the case of periodic forces, we can immediately generalize this result to a sum of impulsive forces. If the oscillator is subjected to a series of blows, of impulse I_r at time t_r, its position may be found by adding together the corresponding set of solutions of the form (2.47). We obtain in this way

$$x(t) = \sum_r G(t-t_r)I_r + \text{transient}, \tag{2.48}$$

where

$$\begin{aligned} G(t-t') &= 0, && t<t'; \\ G(t-t') &= \frac{1}{m\omega}\,\mathrm{e}^{-\gamma(t-t')}\sin\omega(t-t'), && t>t'. \end{aligned} \tag{2.49}$$

The function $G(t-t')$ is called the *Green's function* for the oscillator. It represents the response to a blow of unit impulse delivered at the time t'.

We can now use this result to obtain a solution to (2.30) for an arbitrary force $F(t)$. The method is, in a sense, complementary to Fourier's. Instead of expressing $F(t)$ as a sum of periodic forces, we express it as a sum of impulsive forces. If we divide the time scale into small intervals Δt, then the impulse delivered to the particle

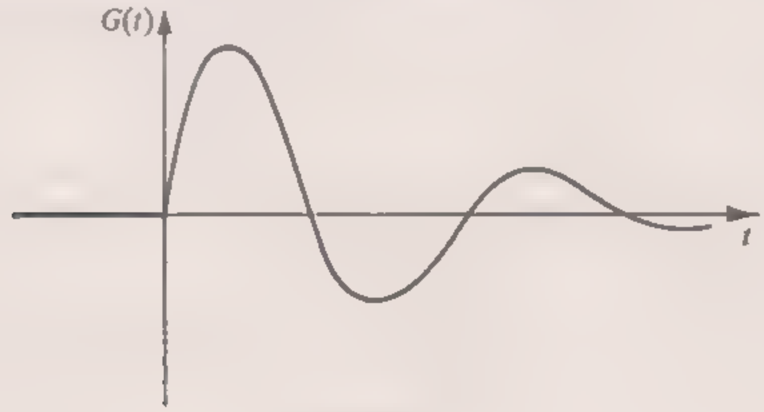


Fig. 2.9

between t and $t+\Delta t$ will be approximately $F(t)\Delta t$. Moreover, if Δt is short enough, it will be a good approximation to suppose that the whole of this impulse is delivered instantaneously at the time t. Thus the solution should be approximately given by (2.48), in which the interval between t_r and t_{r+1} is Δt, and $I_r = F(t_r)\Delta t$. The approximation will improve as Δt is reduced, and become exact in the limit $\Delta t \to 0$. In this limit, the sum in (2.48) goes over into an integral, and we finally obtain

$$x(t) = \int_{t_0}^{t} G(t-t')F(t')\,\mathrm{d}t' + \text{transient}. \tag{2.50}$$

The lower limit t_0 is the initial time at which the initial conditions (determining the arbitrary constants in the transient term) are to be imposed. The upper limit may be taken to be t because $G(t-t')$ vanishes for $t'>t$. (In other words, blows subsequent to t do not affect the position at t.)

The solution (2.50) is very useful in practice, because it is an explicit solution requiring the evaluation of only one integral. It is particularly well adapted to the numerical solution of the problem when $F(t)$ is known numerically.

2.8 Collision Problems

So far we have been discussing the motion of a single particle under a known external force. We now turn to the problem of an isolated system of two bodies moving under a mutual force F which is given in terms of their positions and velocities. The equations of motion are then

$$\begin{aligned} m_1\ddot{x}_1 &= F, \\ m_2\ddot{x}_2 &= -F. \end{aligned} \tag{2.51}$$

We have already seen in Chapter 1 that these equations lead to the law of *conservation of momentum*,

$$p_1 + p_2 = P = \text{constant}. \tag{2.52}$$

According to the relativity principle, F must be a function only of the *relative* distance

$$x = x_1 - x_2$$

and the *relative* velocity

$$\dot{x} = \dot{x}_1 - \dot{x}_2.$$

When it is a function only of x, the force is conservative, and we can introduce a potential energy function $V(x)$ as before. The law of conservation of energy takes the form

$$T + V = E = \text{constant}, \tag{2.53}$$

with

$$T = \tfrac{1}{2}m_1\dot{x}_1{}^2 + \tfrac{1}{2}m_2\dot{x}_2{}^2,$$

as may easily be verified by differentiating (2.53) and using (2.51) and the relation between V and F.

We are particularly interested in collision problems, in which the force is generally small except when the bodies are very close together. The potential function is then a constant (say zero) for large values of x, and rises very sharply for small values. (See Fig. 2.10*a*.) An ideal *impulsive* conservative force corresponds to a potential energy function with a discontinuity or step (Fig. 2.10*b*). So long as the initial kinetic energy is less than the height of the step, the bodies will bounce off one another. (If the kinetic energy exceeds this value, one

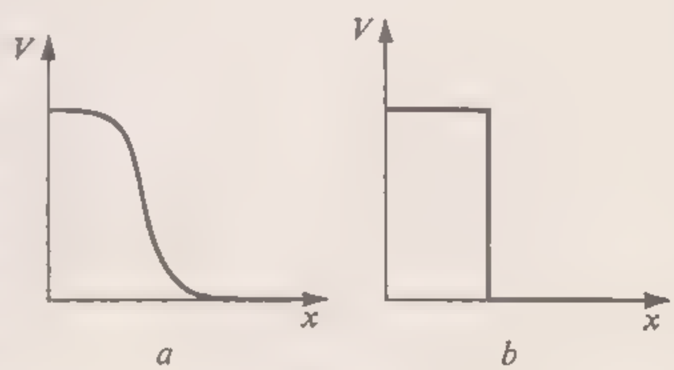


Fig. 2.10

body will pass through the other. For completely impenetrable bodies, the potential step must be infinitely high.) From the law of conservation of energy, we can deduce that the final value of the kinetic energy, when the bodies are again far apart, is the same as the initial value before the collision. If we denote the initial velocities by u_1, u_2, and the final velocities by v_1, v_2, then (see Fig. 2.11)

$$\tfrac{1}{2}m_1v_1^2+\tfrac{1}{2}m_2v_2^2 = \tfrac{1}{2}m_1u_1^2+\tfrac{1}{2}m_2u_2^2.$$

Collisions of this type, in which there is no loss of kinetic energy, are known as *elastic* collisions. They are typical of very hard bodies, like billiard balls.

We saw earlier that a particle which bounces back from a potential barrier emerges with velocity equal (except for sign) to its initial velocity. We can easily derive a similar result for the case of two-body

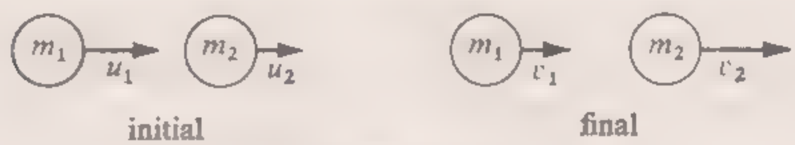


Fig. 2.11

collisions. To do this, we write the energy and momentum conservation equations in the forms

$$\tfrac{1}{2}m_1v_1^2-\tfrac{1}{2}m_1u_1^2 = \tfrac{1}{2}m_2u_2^2-\tfrac{1}{2}m_2v_2^2,$$

$$m_1v_1-m_1u_1 = m_2u_2-m_2v_2.$$

We can then divide the first of these equations by the second, and obtain

$$v_1+u_1 = u_2+v_2$$

or

$$v_2-v_1 = u_1-u_2. \tag{2.54}$$

This shows that the *relative* velocity is just reversed in the collision.

Equation (2.54) and the momentum conservation equation may be solved for the final velocities in terms of the initial velocities. In particular, if the second body is initially at rest ($u_2 = 0$), then

$$\begin{aligned} v_1 &= \frac{m_1-m_2}{m_1+m_2}\,u_1, \\ v_2 &= \frac{2m_1}{m_1+m_2}\,u_1. \end{aligned} \tag{2.55}$$

Note that if the masses are equal, then the first body is brought to rest by the collision, and its velocity transferred to the second body. If $m_1 > m_2$, the first body continues in the same direction, with reduced velocity, whereas if $m_1 < m_2$ it rebounds in the opposite direction. In the limit where m_2 is much larger than m_1, we obtain $v_1 = -u_1$, which agrees with the previous result for a particle rebounding from a fixed potential barrier.

Inelastic Collisions. In practice there is generally some loss of energy in a collision, for instance in the form of heat. In that case, the relative velocity will be reduced in magnitude by the collision. We define the *coefficient of restitution* e in a particular collision by

$$v_2 - v_1 = e(u_1 - u_2). \tag{2.56}$$

The usefulness of this quantity derives from the experimental fact that, for any two given bodies, e is approximately a constant for a wide range of velocities.

The final velocities may again be found from (2.56) and the momentum conservation equation. When $u_2 = 0$, we get

$$v_1 = \frac{m_1 - em_2}{m_1 + m_2} u_1, \qquad v_2 = \frac{(1+e)m_1}{m_1 + m_2} u_1. \tag{2.57}$$

For $e = 1$, these reduce to the relations (2.55) for the case of an elastic collision. At the other extreme, we have the case of very soft bodies which stick together on impact. Then $e = 0$, and the collision is called *perfectly inelastic.*

The energy loss in an inelastic collision is easily evaluated. The initial kinetic energy is $T = \frac{1}{2}m_1 u_1^2$, while the final kinetic energy is $T' = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2$. Substituting for the final velocities from (2.57), we find, after a simple calculation, that the fractional loss of kinetic energy is

$$\frac{T - T'}{T} = \frac{(1 - e^2)m_1}{m_1 + m_2}. \tag{2.58}$$

This shows, incidentally, that e must always be less than one, unless some energy is released, as for example by an explosion.

2.9 Summary

The conservation laws for energy and momentum are among the most important consequences of Newton's laws. They are valid

quite generally for an isolated system, as we shall see in later chapters. For a system which is only part of a larger system, and subjected to external forces, there may be some transfer of energy and momentum from the system to its surroundings (or vice versa). For a single particle, moving on a straight line, the total (kinetic plus potential) energy is conserved if the force is a function only of x. If the force is velocity-dependent, however, there will be some loss of energy to the surroundings.

The importance of the harmonic oscillator, treated in detail in this chapter, lies in the fact that any system with one degree of freedom behaves like a harmonic oscillator when sufficiently near a point of equilibrium. The methods we have discussed for solving the oscillator equation can therefore be applied to a great variety of problems. In particular, the phenomenon of resonance can occur in any system subjected to periodic forces. It occurs when there is a natural frequency of oscillation close to the forcing frequency, and leads, if the damping is small, to oscillations of very large amplitude.

PROBLEMS

1 Discuss the motion of a simple pendulum of length l (supported by a light rigid rod rather than a flexible string) for arbitrary amplitude by drawing the potential energy curve $V(\theta)$. Show that two types of motion (oscillation and continuous revolution) are possible according to the value of the energy.

2 This pendulum is released from rest at a small angle to the upward vertical. Show that the time taken for the angular displacement to increase by a factor of 10 is approximately $(l/g)^{1/2}\ln 20$.* Evaluate this time for a pendulum of period 2 s, and find the angular velocity of the pendulum when it reaches the downward vertical.

3 A particle of mass m moves under a conservative force with potential energy

$$V = \frac{cx}{x^2+a^2},$$

where a and c are positive constants. Find the position of stable equilibrium, and the period of small oscillations about it. If the particle starts from this point with velocity v, find the values of v for which it (i) oscillates, (ii) escapes to $-\infty$ and (iii) escapes to $+\infty$.

4 The natural length of a spring is 10 cm. When a mass is suspended from the end of the spring, the equilibrium length is increased to 12 cm. If the mass is given a blow which starts it moving vertically with velocity 4 cm/s, find the period and amplitude of the resulting oscillations.

5 A pendulum whose period in vacuum is 2 s is placed in a resistive medium. Its amplitude on each swing is observed to be half that of the

* We shall always use the notation $\ln x$ for the natural logarithm, $\log_e x$.

previous swing. What is its new period? If the pendulum is subjected in turn to periodic forces of equal amplitude with periods of (a) 1 s, (b) 2 s and (c) 2·5 s, find the ratios of the amplitudes of the forced oscillations.

6 Write down the solution to the oscillator equation for $\omega_0 > \gamma$ if the oscillator starts from $x = 0$ with velocity v. Show that, as ω_0 is reduced to the critical value $\omega_0 = \gamma$, the solution tends to the corresponding solution for the critically damped oscillator.

7 Solve the problem of an oscillator under a simple periodic force by the Green's function method, and verify that this reproduces the solution of §2.6. (To perform the integration, write $\sin\omega t$ as $(e^{i\omega t} - e^{-i\omega t})/2i$.)

8 For an oscillator under a periodic force, calculate the rate at which the force does work. Show that the average rate is $P = m\gamma\omega_1{}^2a_1{}^2$, and hence verify that it is equal to the average rate at which energy is dissipated against the resistive force. Show that the power P is a maximum, as a function of ω_1, at $\omega_1 = \omega_0$, and find the values of ω_1 for which P has half its maximum value.

9 Find the average value $\bar{E}$ of the total energy of an oscillator under a periodic force. If W is the work done against friction in one period, show that when $\omega_1 = \omega_0$ the ratio $W/\bar{E}$ is related to the quality factor by $W/\bar{E} = 2\pi/Q$.

10 Three perfectly elastic spheres of masses 5 g, 1 g, 5 g are arranged in that order on a straight line. Initially the middle one is moving with velocity 27 cm/s, and the others are at rest. Find how many collisions take place in the subsequent motion, and verify that the final value of the kinetic energy is equal to the initial value.

11 A ball is dropped from height h and bounces. The coefficient of restitution at each bounce is e. Show that it finally comes to rest after a time

$$\frac{1+e}{1-e}\left(\frac{2h}{g}\right)^{1/2}.$$

12 A particle moving under a conservative force oscillates between x_1 and x_2. Show that the period of oscillation is

$$\tau = 2\int_{x_1}^{x_2}\left(\frac{m}{2[V(x_2)-V(x)]}\right)^{1/2} \mathrm{d}x.$$

In particular, if $V = \frac{1}{2}m\omega_0{}^2(x^2 - bx^4)$, show that the period for oscillations of amplitude a is

$$\tau = \frac{2}{\omega_0}\int_{-a}^{a}\frac{\mathrm{d}x}{(a^2-x^2)^{1/2}[1-b(a^2+x^2)]^{1/2}}.$$

Using the binomial theorem to expand in powers of b, show that for small amplitude the period is approximately

$$\tau \approx \frac{2\pi}{\omega_0}(1+\tfrac{3}{4}ba^2).$$

13 Use the result of the preceding question to obtain an estimate of the correction factor to the period of a pendulum when its angular amplitude is 30°.

$$\left(\text{Write } \cos\theta \approx 1-\frac{\theta^2}{2}+\frac{\theta^4}{24}\right).$$

14 An oscillator with free oscillation period τ is critically damped and subjected to a periodic force with the 'saw-tooth' form

$$F(t) = c(t-n\tau), \qquad (n-\tfrac{1}{2})\tau<t<(n+\tfrac{1}{2})\tau,$$

for each integer n (c is a constant). Find the ratios of the amplitudes of oscillation at the angular frequencies $2\pi n/\tau$.

Energy and Angular Momentum Chapter 3

In this chapter, we shall generalize the discussion of Chapter 2 to the case of three-dimensional motion. Throughout this chapter, we shall be concerned with the problem of a particle moving under a known external force **F**.

3.1 Energy; Conservative Forces

The *kinetic energy* of a particle of mass m free to move in three dimensions is defined to be

$$T = \tfrac{1}{2}m\dot{\mathbf{r}}^2 = \tfrac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2). \tag{3.1}$$

The rate of change of the kinetic energy is, therefore,

$$\dot{T} = m(\dot{x}\ddot{x}+\dot{y}\ddot{y}+\dot{z}\ddot{z}) = m\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}} = \dot{\mathbf{r}}\cdot\mathbf{F}, \tag{3.2}$$

by the equation of motion (1.1). The change in kinetic energy in a time interval $\mathrm{d}t$ during which the particle moves a (vector) distance $\mathrm{d}\mathbf{r}$ is then

$$\mathrm{d}T = \mathrm{d}W, \tag{3.3}$$

with

$$\mathrm{d}W = \mathbf{F}\cdot\mathrm{d}\mathbf{r} = F_x\,\mathrm{d}x+F_y\,\mathrm{d}y+F_z\,\mathrm{d}z. \tag{3.4}$$

This is the three-dimensional expression for the *work* done by the force **F** in the displacement $\mathrm{d}\mathbf{r}$. Note that it is equal to the distance travelled $|\mathrm{d}\mathbf{r}|$ multiplied by the component of **F** in the direction of the displacement.

One might think at first sight that a conservative force in three dimensions should be defined to be a force $\mathbf{F}(\mathbf{r})$ depending only on the position $\mathbf{r}$ of the particle. This is, however, not sufficient to ensure the existence of a law of conservation of energy, which is the essential feature of a conservative force. We require that

$$T+V = E = \text{constant}, \tag{3.5}$$

where T is given by (3.1), and the *potential energy* V is a function of position, $V(\mathbf{r})$. Now, the rate of change of the potential energy function is given by

$$\dot{V}(\mathbf{r}) = \frac{\partial V}{\partial x}\,\dot{x}+\frac{\partial V}{\partial y}\,\dot{y}+\frac{\partial V}{\partial z}\,\dot{z},$$

or, in terms of the *gradient* of V

$$\nabla V = \mathbf{i}\frac{\partial V}{\partial x}+\mathbf{j}\frac{\partial V}{\partial y}+\mathbf{k}\frac{\partial V}{\partial z},$$

(see Appendix A (A.21)), by

$$\dot{V} = \dot{\mathbf{r}}\cdot\nabla V. \tag{3.6}$$

Thus, differentiating (3.5), and using (3.2) and (3.6) for $\dot{T}$ and $\dot{V}$, we obtain

$$\dot{\mathbf{r}}\cdot(\mathbf{F}+\nabla V) = 0.$$

Since this must hold for any velocity of the particle, we require

$$\mathbf{F}(\mathbf{r}) = -\nabla V(\mathbf{r}). \tag{3.7}$$

This is the analogue of (2.5) for one-dimensional forces. In terms of components, it reads

$$F_x = -\frac{\partial V}{\partial x}, \qquad F_y = -\frac{\partial V}{\partial y}, \qquad F_z = -\frac{\partial V}{\partial z}. \tag{3.8}$$

It is not hard to find the necessary and sufficient conditions on the force $\mathbf{F}(\mathbf{r})$ to ensure the existence of a potential energy function $V(\mathbf{r})$ satisfying (3.7). Any vector function $\mathbf{F}(\mathbf{r})$ of the form (3.7) obeys the relation

$$\nabla\wedge\mathbf{F} = \mathbf{0}, \tag{3.9}$$

that is, its *curl* vanishes (see Appendix A (A.26)). For instance, the z component of (3.9) is

$$\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y} = 0,$$

and this is true because of (3.8) and the symmetry of the second partial derivative:

$$\frac{\partial^2 V}{\partial x\,\partial y} = \frac{\partial^2 V}{\partial y\,\partial x}.$$

Thus (3.9) is a necessary condition for the force $\mathbf{F}(\mathbf{r})$ to be conservative. It can be shown (see Appendix A) that it is also a sufficient condition. More specifically, if the force $\mathbf{F}(\mathbf{r})$ satisfies this condition, then the work done by the force in a displacement from $\mathbf{r}_0$ to $\mathbf{r}$ is independent of the path chosen between these points. Therefore, we can define the potential energy $V(\mathbf{r})$ in terms of the force by a relation analogous to (2.4).

3.2 Moments; Angular Momentum

The *moment about the origin* of a force $\mathbf{F}$ acting on a particle at position $\mathbf{r}$ is defined to be the vector product

$$\mathbf{G} = \mathbf{r} \wedge \mathbf{F}. \tag{3.10}$$

The components of the vector $\mathbf{G}$ are the *moments about the x-, y-, z-axes*,

$$\begin{aligned} G_x &= yF_z - zF_y, \\ G_y &= zF_x - xF_z, \\ G_z &= xF_y - yF_x. \end{aligned} \tag{3.11}$$

The direction of the vector $\mathbf{G}$ is that of the normal to the plane of $\mathbf{r}$ and $\mathbf{F}$. It may be regarded as defining the axis about which the force $\mathbf{F}$ tends to rotate the particle. The magnitude of $\mathbf{G}$ is

$$G = rF \sin\theta = bF,$$

where θ is the angle between $\mathbf{r}$ and $\mathbf{F}$, and b is the perpendicular distance from the origin to the line of action of the force. (See Fig. 3.1.)

Correspondingly, we define the vector *angular momentum* (sometimes called *moment of momentum*) *about the origin* of a particle at position $\mathbf{r}$, and moving with momentum $\mathbf{p}$, to be

$$\mathbf{J} = \mathbf{r} \wedge \mathbf{p} = m\mathbf{r} \wedge \dot{\mathbf{r}}. \tag{3.12}$$

Its components, the angular momenta *about the x-, y-, z-axes*, are

$$\begin{aligned} J_x &= m(y\dot{z} - z\dot{y}), \\ J_y &= m(z\dot{x} - x\dot{z}), \\ J_z &= m(x\dot{y} - y\dot{x}). \end{aligned} \tag{3.13}$$

The rate of change of the angular momentum $\mathbf{J}$ is

$$\dot{\mathbf{J}} = m\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r} \wedge \dot{\mathbf{r}}) = m(\dot{\mathbf{r}} \wedge \dot{\mathbf{r}} + \mathbf{r} \wedge \ddot{\mathbf{r}}). \tag{3.14}$$

(Recall that the usual rule for differentiating a product applies also to vector products, *provided* that the order of the two factors is preserved.) Now, the first term in (3.14) is zero, because it is the vector product of a vector with itself. The second term is simply $\mathbf{r} \wedge \mathbf{F} = \mathbf{G}$. Thus we obtain the important result that the rate of change of angular momentum is equal to the moment of the applied force,

$$\dot{\mathbf{J}} = \mathbf{G}. \tag{3.15}$$

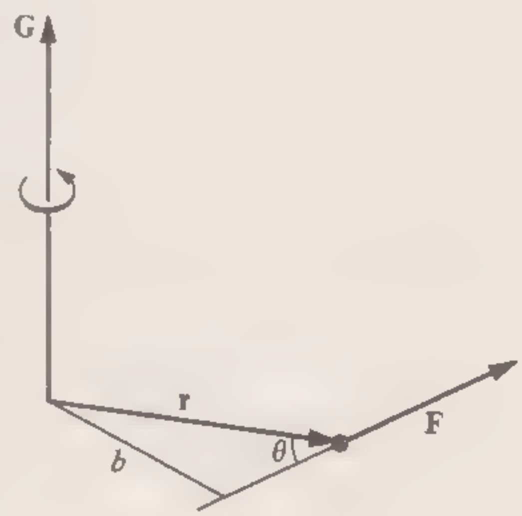


Fig. 3.1

This should be compared with the equation $\dot{\mathbf{p}} = \mathbf{F}$ for the rate of change of linear momentum.

Since the definition of the vector product (Appendix A, §A.2) depends on the choice of a right-hand screw convention, the directions of the vectors **G** and **J** also depend on this convention, and would be reversed by changing to a left-hand convention. A vector of this type is known as an *axial* vector. It is to be contrasted with an ordinary, or *polar*, vector, whose direction is defined independently of any choice of convention. Axial vectors are often associated with rotation about an axis. What is specified physically is not the direction along the axis, but the sense of rotation about it. (See Fig. 3.1.)

3.3 Central Forces; Conservation of Angular Momentum

An external force **F** is said to be *central* if it is always directed towards or away from a fixed point, called the *centre* of force. If we choose the origin to be at this centre, this means that **F** is always parallel to the position vector **r**. Since the vector product of two parallel vectors is zero, the condition for a force **F** to be central is that its moment about the centre should vanish:

$$\mathbf{G} = \mathbf{r} \wedge \mathbf{F} = \mathbf{0}. \tag{3.16}$$

From (3.15) it follows that when the force is central, the angular momentum is a constant:

$$\mathbf{J} = \text{constant.} \tag{3.17}$$

This is the law of *conservation of angular momentum* in its simplest form. It will be useful to discuss in some detail the physical significance of this law. It really contains two statements: that the direction of **J** is constant, and that its magnitude is constant. Now, the direction of **J** is that of the normal to the plane of **r** and **v**. Hence, the statement that this direction is fixed implies that **r** and **v** must always lie in a fixed plane. In other words, the motion of the particle is confined to the plane containing the initial position vector and velocity vector. This is obvious physically; for, since the force is central, it has no component perpendicular to this plane, and, since the normal component of the velocity is initially zero, it must always remain zero.

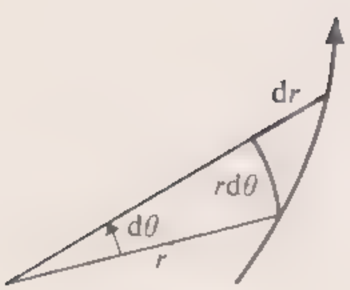


Fig. 3.2

To understand the meaning of the second part of the law, the constancy of the magnitude of **J**, it is convenient to introduce polar co-ordinates r and θ in the plane of the motion. In a short time interval dt, in which the co-ordinates change by amounts dr and dθ, the

distances travelled in the radial and transverse directions are dr and $r\mathrm{d}\theta$, respectively. (See Fig. 3.2.) Thus the radial and transverse components of the velocity are

$$v_r = \dot{r}, \qquad v_\theta = r\dot{\theta}. \tag{3.18}$$

The magnitude of the angular momentum is then

$$J = mrv_\theta = mr^2\dot{\theta}. \tag{3.19}$$

It is now easy to find a geometrical interpretation of the statement that J is a constant. We note that when the angle θ changes by an amount dθ, the radius vector sweeps out an area

$$\mathrm{d}A = \tfrac{1}{2}r^2\,\mathrm{d}\theta.$$

Thus the rate of sweeping out area is

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \tfrac{1}{2}r^2\,\dot{\theta} = \frac{J}{2m} = \text{constant}. \tag{3.20}$$

This law is generally referred to as Kepler's second law of planetary motion, though, as we have seen, it applies more generally to motion under any central force. It applies even if the force is non-conservative—for example, to a particle attached to the end of a string which is gradually being wound in to the centre. An equivalent statement of the law, which follows at once from (3.19), is that the transverse component of velocity v_θ varies inversely with the radial distance r.

When the force is both central and conservative, the two conservation laws (3.5) and (3.17) together give us a great deal of information about the motion of the particle. We shall discuss this case in the next chapter.

3.4 Polar Co-ordinates

In problems with particular symmetries, it is often convenient to use non-Cartesian co-ordinates. In particular, in the case of axial or spherical symmetry, we may use cylindrical polar co-ordinates ρ, φ, z, or spherical polar co-ordinates r, θ, φ. These are related to the Cartesian co-ordinates by

from § 3-6 Hamilton's Principle p47

$$\begin{aligned} x &= \rho\cos\varphi = r\sin\theta\cos\varphi, \\ y &= \rho\sin\varphi = r\sin\theta\sin\varphi, \\ z &= z \qquad\;\;\, = r\cos\theta. \end{aligned} \tag{3.21}$$

(See Fig. 3.3.) The relation between the two is clearly

$$
\begin{aligned}
z &= r\cos\theta, \\
\rho &= r\sin\theta, \\
\varphi &= \varphi.
\end{aligned}
\tag{3.22}
$$

It is easy to find the components of velocity in the various co-ordinate directions, analogous to (3.18). In the cylindrical polar case, the co-ordinates ρ and φ are just plane polar co-ordinates in the xy-plane, and we obviously have

$$v_\rho = \dot{\rho}, \qquad v_\varphi = \rho\dot{\varphi}, \qquad v_z = \dot{z}. \tag{3.23}$$

In the spherical polar case, a curve along which θ varies, while r and φ are fixed, is a circle of radius r (corresponding on the earth to a meridian of longitude). A curve along which only φ varies is a circle of radius $\rho = r\sin\theta$ (corresponding to a parallel of latitude). Thus the elements of length in the three co-ordinate directions are $\mathrm{d}r$, $r\mathrm{d}\theta$ and $r\sin\theta\,\mathrm{d}\varphi$, and we get

$$v_r = \dot{r}, \qquad v_\theta = r\dot{\theta}, \qquad v_\varphi = r\sin\theta\,\dot{\varphi}. \tag{3.24}$$

It will be useful to have expressions for the kinetic energy in terms of polar co-ordinates. These co-ordinates, though curvilinear, are still *orthogonal* in the sense that the three co-ordinate directions at each point are mutually perpendicular. Thus we can write, for example,

$$\mathbf{v}^2 = v_\rho{}^2 + v_\varphi{}^2 + v_z{}^2.$$

(This would not be true if the co-ordinates were oblique.) Hence from (3.23) and (3.24) we find

$$T = \tfrac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2) \tag{3.25}$$

and

$$T = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2). \tag{3.26}$$

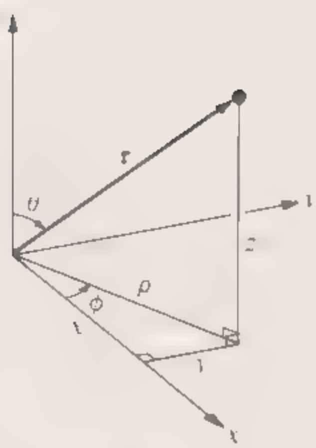


Fig. 3.3

Note the characteristic difference between these expressions and the corresponding one for the case of Cartesian co-ordinates—here T depends not only on the time derivatives of the co-ordinates, but also on the co-ordinates themselves.

We could now go on and find expressions for the acceleration in terms of these co-ordinates, and hence write down equations of motion. While it is not particularly hard to do this directly in these simple cases, it is very convenient to have a general prescription for writing down the equations of motion in arbitrary co-ordinates. This can in fact be done, by a method due to Lagrange, as soon as

we have found the expression for the kinetic energy, like (3.25) or (3.26).

The simplest—though not the only—way of deriving Lagrange's equations is to use what is known as a variational principle. This is a principle which states that some quantity has a minimum value, or, more generally, a stationary value. Principles of this kind are used in many branches of physics, particularly in quantum mechanics and in optics. It may therefore be useful, before discussing the particular principle we shall require, to consider the general kind of problem in which such methods are used. This we shall do in the next section.

3.5 The Calculus of Variations

from § 11·6 The Stretched String. That the assumptions about the plane may be specified.

It will be helpful to begin by discussing a very simple example. Let us ask the question: what is the shortest path between two given points in a plane? Of course we know the answer already, but the method we shall use to derive it can also be applied to less trivial examples—for instance, to find the shortest path between two points on a curved surface.

Suppose the two points are (x_0, y_0) and (x_1, y_1). Any curve joining them is represented by an equation

The coordinate system taken entirely within the plane.

$$y = y(x),$$

such that the function $y(x)$ satisfies the boundary conditions

$$y(x_0) = y_0, \qquad y(x_1) = y_1. \tag{3.27}$$

Consider two neighbouring points on this curve. The distance dl between them is given by

$$dl = (dx^2 + dy^2)^{1/2} = (1 + y'^2)^{1/2}\, dx,$$

where $y' = dy/dx$. Thus the total length of the curve is

$$l = \int_{x_0}^{x_1} (1 + y'^2)^{1/2}\, dx. \tag{3.28}$$

quoted in § 11·6.

The problem, therefore, is to find that function $y(x)$, subject to the conditions (3.27), which will make this integral a minimum.

This problem differs from the usual kind of minimum-value problem in that what we have to vary is not a single variable or set of variables, but a *function* $y(x)$. However, we can still apply the same

criterion: when the integral has a minimum value, it must be unchanged to first order by making a small variation in the function $y(x)$. (We shall not be concerned with the problem of distinguishing maxima from minima. All we shall do is to find the stationary values.)

More generally, we may be interested in finding the stationary values of an integral of the form

$$I = \int_{x_0}^{x_1} f(y, y')\,\mathrm{d}x, \tag{3.29}$$

where $f(y, y')$ is a specified function of y and its first derivative. We shall solve this general problem, and then apply the result to the integral (3.28). Consider a small variation $\delta y(x)$ in the function $y(x)$, subject to the condition that the values of y at the end-points are unchanged (see Fig. 3.4):

$$\delta y(x_0) = 0, \qquad \delta y(x_1) = 0. \tag{3.30}$$

To first order, the variation in $f(y, y')$ is

$$\delta f = \frac{\partial f}{\partial y}\,\delta y + \frac{\partial f}{\partial y'}\,\delta y',$$

where

$$\delta y' = \frac{\mathrm{d}}{\mathrm{d}x}\,\delta y.$$

Thus the variation of the integral I is

$$\delta I = \int_{x_0}^{x_1} \left[\frac{\partial f}{\partial y}\,\delta y + \frac{\partial f}{\partial y'}\,\frac{\mathrm{d}}{\mathrm{d}x}\,\delta y\right] \mathrm{d}x.$$

In the second term, we may integrate by parts. The integrated term, namely

$$\left[\frac{\partial f}{\partial y'}\,\delta y\right]_{x_0}^{x_1}$$

vanishes at the limits because of the conditions (3.30). Hence we obtain

$$\delta I = \int_{x_0}^{x_1} \left[\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)\right] \delta y(x)\,\mathrm{d}x. \tag{3.31}$$

Now, in order that I should be stationary, this variation δI must vanish for an *arbitrary* small variation $\delta y(x)$. This is only possible if the integrand vanishes identically. Thus we require

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) = 0. \tag{3.32}$$

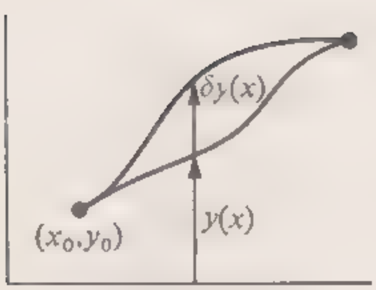


Fig. 3.4

This is known as the Euler–Lagrange equation. It is in general a second-order differential equation for the function $y(x)$, whose solution contains two arbitrary constants that may be determined from the known values of y at x_0 and x_1.

We can now solve the problem we started with. In that case, comparing (3.28) and (3.29), we have to choose

$$f = (1+y'^2)^{1/2},$$

and therefore

$$\frac{\partial f}{\partial y} = 0, \qquad \frac{\partial f}{\partial y'} = \frac{y'}{(1+y'^2)^{1/2}}.$$

Thus the Euler–Lagrange equation (3.32) reads

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{y'}{(1+y'^2)^{1/2}}\right] = 0.$$

This equation states that the expression inside the brackets is a constant, and therefore that y' is a constant. Its solutions are thus the straight lines

$$y = ax+b.$$

Thus we have proved that the shortest path between two points is a straight line. The constants a and b are of course fixed by the conditions (3.27).

So far, we have used x as the independent variable, but in the applications we consider later we shall be concerned instead with functions of the time t. It is easy to generalize the discussion to the case of a function f of n variables $q_1, q_2, \ldots, q_n$, and their time derivatives $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n$. In order that the integral

$$I = \int_{t_0}^{t_1} f(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n)\,\mathrm{d}t$$

be stationary, it must be unchanged to first order by a variation in any one of the functions $q_i(t)$ $(i = 1, 2, \ldots, n)$, subject to the conditions $\delta q_i(t_0) = \delta q_i(t_1) = 0$. Thus we require the n Euler–Lagrange equations

$$\frac{\partial f}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{q}_i}\right) = 0, \qquad i = 1, 2, \ldots, n. \tag{3.33}$$

These n second-order differential equations determine the n functions $q_i(t)$ to within $2n$ arbitrary constants of integration.

from p 134.

3.6 Hamilton's Principle; Lagrange's Equations

We shall now show that the equations of motion for a particle moving under a conservative force can be written as the Euler–Lagrange equations corresponding to a suitable integral. We define the Lagrangian function

$$L = T - V = \tfrac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) - V(x, y, z). \tag{3.34}$$

(Note the minus sign—this is *not* the total energy.) Its derivatives are

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} = p_x, \qquad \frac{\partial L}{\partial x} = -\frac{\partial V}{\partial x} = F_x, \tag{3.35}$$

with similar expressions for the y and z components. Thus the equation of motion

$$\dot{p}_x = F_x$$

can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}. \tag{3.36}$$

But this has precisely the form of an Euler–Lagrange equation for the integral

$$I = \int_{t_0}^{t_1} L\,\mathrm{d}t. \tag{3.37}$$

This is known as the *action integral*. Thus we have obtained Hamilton's principle of least action: the action integral I is stationary under arbitrary variations δx, δy, δz which vanish at the limits of integration t_0 and t_1.

→ § 11.2 Lagrange's Equations p167 →

The importance of this principle lies in the fact that it can immediately be applied to any set of co-ordinates. If, instead of using Cartesian co-ordinates x, y, z, we employ a set of curvilinear co-ordinates q_1, q_2, q_3, then we can express the Lagrangian function $L = T - V$ in terms of q_1, q_2, q_3, and their time derivatives $\dot{q}_1$, $\dot{q}_2$, $\dot{q}_3$. The action integral (3.37) must then be stationary with respect to arbitrary variations δq_1, δq_2, δq_3, subject to the condition that the variations are zero at the limits t_0 and t_1. Thus we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}, \qquad i = 1, 2, 3. \tag{3.38}$$

These are *Lagrange's equations*. They are the equations of motion in terms of the co-ordinates q_1, q_2, q_3. By analogy with (3.36), we may

define the *generalized momenta*

$$p_i = \frac{\partial L}{\partial \dot{q}_i}, \tag{3.39}$$

and the *generalized forces*

$$F_i = \frac{\partial L}{\partial q_i}, \tag{3.40}$$

so that the equations (3.38) read

$$\dot{p}_i = F_i. \tag{3.41}$$

It must be emphasized, however, that in general the p_i and F_i are *not* components of the momentum vector $\mathbf{p}$ and the force vector $\mathbf{F}$—we shall see this explicitly in the examples considered below. → p167

Let us now apply these equations to the polar co-ordinates discussed in §3.4. In the cylindrical polar case, the derivatives of the kinetic energy function (3.25) are → p41 from end.

$$\frac{\partial T}{\partial \dot{\rho}} = m\dot{\rho}, \qquad \frac{\partial T}{\partial \dot{\varphi}} = m\rho^2\dot{\varphi}, \qquad \frac{\partial T}{\partial \dot{z}} = m\dot{z},$$

$$\frac{\partial T}{\partial \rho} = m\rho\dot{\varphi}^2, \qquad \frac{\partial T}{\partial \varphi} = 0, \qquad \frac{\partial T}{\partial z} = 0.$$

Hence Lagrange's equations (3.38) take the form

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{\rho}) &= m\rho\dot{\varphi}^2 - \frac{\partial V}{\partial \rho}, \\ \frac{\mathrm{d}}{\mathrm{d}t}(m\rho^2\dot{\varphi}) &= -\frac{\partial V}{\partial \varphi}, \\ \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{z}) &= -\frac{\partial V}{\partial z}. \end{aligned} \tag{3.42}$$

The generalized momenta $m\dot{\rho}$ and $m\dot{z}$ corresponding to the co-ordinates ρ and z are the components of the momentum vector $\mathbf{p}$ in the ρ and z directions. However, the generalized momentum $m\rho^2\dot{\varphi}$ is an *angular* momentum; in fact the angular momentum J_z about the z-axis. It is easy to see that, whenever q_i is an angle, the corresponding generalized momentum p_i has the dimensions of angular, rather than linear, momentum. Note that when the potential energy is independent of φ, so that there is rotational symmetry about the z-axis, then the second of the equations (3.42) expresses the fact that the component J_z of angular momentum is conserved.

Proceeding in a similar way, we can write down the equations of motion in spherical polars in terms of the derivatives of the kinetic energy function (3.26). We find

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{r}) &= mr(\dot{\theta}^2+\sin^2\theta\,\dot{\varphi}^2)-\frac{\partial V}{\partial r}\\ \frac{\mathrm{d}}{\mathrm{d}t}(mr^2\dot{\theta}) &= mr^2\sin\theta\cos\theta\,\dot{\varphi}^2-\frac{\partial V}{\partial\theta}, \qquad (3.43)\\ \frac{\mathrm{d}}{\mathrm{d}t}(mr^2\sin^2\theta\,\dot{\varphi}) &= -\frac{\partial V}{\partial\varphi}.\end{aligned}$$

Once again, the generalized momentum $mr^2\sin^2\theta\,\dot{\varphi}$ is the component of angular momentum J_z about the z-axis, and is a constant if V is independent of φ. The generalized momentum $mr^2\dot{\theta}$ also has the dimensions of angular momentum. Since it is equal to the radial distance r, multiplied by the θ component of momentum, $mr\dot{\theta}$, it is in fact the φ component of angular momentum, J_φ. Note, however, that even when V is independent of both angular variables θ and φ, J_φ is not in general a constant. This is because the φ direction is not a fixed direction in space, like the z-axis, but changes according to the position of the particle. Thus, even when $\mathbf{J}$ is a fixed vector, the component J_φ may vary.

Before concluding this section, it will be useful to discuss the special case of a *central* conservative force, which is treated in detail in the following chapter. In spherical polars, the relation (3.7) between force and potential energy becomes (see Appendix A (A.42))

$\underline{F}(\vec{r}) = -\nabla V(\vec{r})$

$$F_r = -\frac{\partial V}{\partial r}, \qquad F_\theta = -\frac{1}{r}\frac{\partial V}{\partial\theta}, \qquad F_\varphi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial\varphi}.$$

Thus the force $\mathbf{F}$ will be purely radial if, and only if, V is independent of the angular co-ordinates θ and φ. Then we have

$$\mathbf{F} = -\hat{\mathbf{r}}\frac{\mathrm{d}V(r)}{\mathrm{d}r}. \qquad (3.44)$$

This shows that the magnitude of a central conservative force depends only on r, and not on θ or φ.

$(3.17) \leftarrow (3.16) \leftarrow \underline{F} \parallel \vec{r}$

"We know from the discussion of §3.3 that the angular momentum vector $\mathbf{J}$ is in this case a constant, and that the motion is confined to a plane. The equations of motion are considerably simplified by an appropriate choice of the polar axis $\theta = 0$. If it is so chosen that $\dot{\varphi} = 0$ initially, then by the third of equations (3.43), $\dot{\varphi}$ will always

be zero. The motion is then in a plane, φ = constant, through the polar axis, in which r and θ play the role of plane polar co-ordinates. The angular momentum $J = J_\varphi = mr^2\dot{\theta}$ is then a constant, by the second of equations (3.43).

Similarly, if the particle is originally moving in the 'equatorial' plane $\theta = \frac{1}{2}\pi$, with $\dot{\theta} = 0$, then from the second equation it follows that θ is always $\frac{1}{2}\pi$. In this case, the plane polar co-ordinates are r and φ, and the angular momentum is $J = J_z = mr^2\dot{\varphi}$.

3.7 Summary

The rate of change of kinetic energy of a particle is the rate at which the force does work, $\dot{T} = \dot{\mathbf{r}}\cdot\mathbf{F}$. The force is conservative if it has the form $\mathbf{F} = -\nabla V(\mathbf{r})$, and in that case the total energy $T+V$ is a constant. The condition for the existence of such a function V is that $\mathbf{F}$ should be a function of position such that $\nabla \wedge \mathbf{F}(\mathbf{r}) = \mathbf{0}$.

The rate of change of angular momentum of a particle is equal to the moment of the force, $\dot{\mathbf{J}} = \mathbf{r} \wedge \mathbf{F}$. When the force is central, the angular momentum is conserved. Then the motion is confined to a plane, and the rate of sweeping out area in this plane is constant.

The use of these conservation laws greatly simplifies the treatment of any problem involving central or conservative forces. When the force is both central and conservative, they provide all the information we need to determine the motion of the particle, as we shall see in the following chapter.

Lagrange's equations are of great importance in advanced treatments of mechanics (and also in quantum mechanics). We have seen that they can be used to write down equations of motion in any system of co-ordinates, as soon as we have found the expressions for the kinetic energy and potential energy. In later chapters we shall see that the method can readily be extended to more complicated systems than a single particle.

PROBLEMS

1 Find which of the following forces are conservative, and for those that are find the corresponding potential energy function (a and b are constants, and $\mathbf{a}$ is a constant vector):

(i) $F_x = ax+by^2, \quad F_y = az+2bxy, \quad F_z = ay+bz^2;$
(ii) $F_x = ay, \quad F_y = az, \quad F_z = ax;$
(iii) $F_r = 2ar\sin\theta\sin\varphi, \quad F_\theta = ar\cos\theta\sin\varphi, \quad F_\varphi = ar\cos\varphi;$
(iv) $\mathbf{F} = \mathbf{a} \wedge \mathbf{r};$
(v) $\mathbf{F} = \mathbf{a}r;$
(vi) $\mathbf{F} = \mathbf{a}(\mathbf{a}\cdot\mathbf{r}).$

2 Evaluate the force corresponding to the potential energy function $V(\mathbf{r}) = cz/r^3$, where c is a constant. Write your answer in vector notation, and also in spherical polars, and verify that it satisfies $\nabla \wedge \mathbf{F} = \mathbf{0}$.

3 A particle of mass m is attached to the end of a light string of length l. The other end of the string is passed through a small hole, and is slowly pulled through it. The particle is originally spinning round the hole with angular velocity ω. Find the angular velocity when the length of the string has been reduced to $\frac{1}{2}l$. Find also the tension in the string when its length is r, and verify that the increase in kinetic energy is equal to the work done by the force pulling the string through the hole. (Neglect gravity).

4 The position on the surface of a cone of semivertical angle α is specified by the distance r from the vertex, and the azimuth angle φ about the axis. Show that the shortest path along the surface between two given points is specified by a function $r(\varphi)$ obeying the equation

$$r\frac{\mathrm{d}^2r}{\mathrm{d}\varphi^2}-2\left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2-r^2\sin^2\alpha = 0.$$

Verify that this equation is satisfied if

$$r = r_0 \sec[(\varphi-\varphi_0)\sin\alpha].$$

5 Find the kinetic energy function, and the equations of motion for a particle of mass m in terms of the following pairs of co-ordinates in a plane:

(i) parabolic co-ordinates $\xi = r+x$ and $\eta = r-x$;
(ii) oblique co-ordinates x and $\eta = ax+by$ $(b \neq 0)$;
(iii) elliptic co-ordinates λ, θ defined by

$$x = (a^2+\lambda^2)^{1/2}\cos\theta, \qquad y = \lambda\sin\theta, \qquad (\lambda \geqslant 0, 0 \leqslant \theta \leqslant 2\pi).$$

Central Conservative Forces — *Chapter 4*

The most important forces of this type are the gravitational and electrostatic forces, which obey the 'inverse square law'. Much of this chapter will therefore be devoted to this special case. We begin, however, by discussing a problem for which it is particularly easy to solve the equations of motion—the three-dimensional analogue of the harmonic oscillator discussed in Chapter 2.

§2.2 Motion near Equilibrium; the Harmonic Oscillator

4.1 The Isotropic Harmonic Oscillator

We consider a particle moving under a central restoring force proportional to its distance from the origin,

$$\mathbf{F} = -k\mathbf{r}. \tag{4.1}$$

The corresponding equation of motion is

$$m\ddot{\mathbf{r}}+k\mathbf{r} = \mathbf{0},$$

or, in terms of components,

$$m\ddot{x}+kx = 0, \qquad m\ddot{y}+ky = 0, \qquad m\ddot{z}+kz = 0.$$

The equation of motion for each co-ordinate is thus identical with the equation (2.8) for the simple harmonic oscillator. This oscillator is called *isotropic* because all directions are equivalent. The *anisotropic* oscillator (which we shall not discuss) is described by similar equations, but with different constants in the three equations. The general solution is again given by (2.13):

$$x = c_x \cos \omega t+d_x \sin \omega t,$$

$$y = c_y \cos \omega t+d_y \sin \omega t,$$

$$z = c_z \cos \omega t+d_z \sin \omega t,$$

where $\omega^2 = k/m$. In vector notation, the solution is

$$\mathbf{r} = \mathbf{c} \cos \omega t+\mathbf{d} \sin \omega t. \tag{4.2}$$

Clearly, the motion is periodic, with period $\tau = 2\pi/\omega$.

As in the linear case, the arbitrary constant vectors $\mathbf{c}$ and $\mathbf{d}$ are determined by the initial conditions. If at $t = 0$ the particle is at $\mathbf{r}_0$, and moving with velocity $\mathbf{v}_0$, then

$$\mathbf{c} = \mathbf{r}_0, \qquad \mathbf{d} = \mathbf{v}_0/\omega. \tag{4.3}$$

We can easily verify the conservation laws for angular momentum and energy. From (4.2) it is clear that $\mathbf{r}$ always lies in the plane of $\mathbf{c}$ and $\mathbf{d}$ (or $\mathbf{r}_0$ and $\mathbf{v}_0$), so that the direction of $\mathbf{J}$ is fixed. Using

$$\dot{\mathbf{r}} = -\omega\mathbf{c}\sin\omega t + \omega\mathbf{d}\cos\omega t, \tag{4.4}$$

we find explicitly

$$\mathbf{J} = m\mathbf{r}\wedge\dot{\mathbf{r}} = m\omega\mathbf{c}\wedge\mathbf{d} = m\mathbf{r}_0\wedge\mathbf{v}_0, \tag{4.5}$$

which is obviously a constant.

The potential energy function $V(r)$ corresponding to the force (4.1) is

$$V = \tfrac{1}{2}kr^2 = \tfrac{1}{2}k(x^2+y^2+z^2). \tag{4.6}$$

Thus, evaluating the energy

$$E = \tfrac{1}{2}m\dot{\mathbf{r}}^2 + \tfrac{1}{2}k\mathbf{r}^2,$$

we find

$$E = \tfrac{1}{2}k(\mathbf{c}^2+\mathbf{d}^2) = \tfrac{1}{2}k\mathbf{r}_0{}^2 + \tfrac{1}{2}m\mathbf{v}_0{}^2, \tag{4.7}$$

which is again a constant.

To find the shape of the particle orbit, it is convenient to rewrite (4.2) in a slightly different form. If θ is any fixed angle, we can write

$$\mathbf{r} = \mathbf{a}\cos(\omega t-\theta) + \mathbf{b}\sin(\omega t-\theta), \tag{4.8}$$

where

$$\begin{aligned} \mathbf{c} &= \mathbf{a}\cos\theta - \mathbf{b}\sin\theta, \\ \mathbf{d} &= \mathbf{a}\sin\theta + \mathbf{b}\cos\theta, \end{aligned} \tag{4.9}$$

or, equivalently,

$$\begin{aligned} \mathbf{a} &= \mathbf{c}\cos\theta + \mathbf{d}\sin\theta, \\ \mathbf{b} &= -\mathbf{c}\sin\theta + \mathbf{d}\cos\theta. \end{aligned}$$

We now choose θ so that $\mathbf{a}$ and $\mathbf{b}$ are perpendicular. This requires

$$\mathbf{a}\cdot\mathbf{b} = -(\mathbf{c}^2-\mathbf{d}^2)\sin\theta\cos\theta + \mathbf{c}\cdot\mathbf{d}(\cos^2\theta - \sin^2\theta) = 0,$$

or

$$\tan 2\theta = \frac{2\mathbf{c}\cdot\mathbf{d}}{\mathbf{c}^2-\mathbf{d}^2}.$$

We may then choose our axes so that the x-axis is in the direction of $\mathbf{a}$, and the y-axis in the direction of $\mathbf{b}$. Equation (4.8) then becomes

$$\begin{aligned} x &= a\cos(\omega t-\theta), \\ y &= b\sin(\omega t-\theta), \\ z &= 0. \end{aligned}$$

The equation of the orbit is obtained by eliminating the time from these equations. It is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1, \qquad z = 0. \tag{4.10}$$

This is the well-known equation of an ellipse with centre at the origin, and semi-axes a and b. (See Fig. 4.1.) The vectors $\mathbf{c}$ and $\mathbf{d}$ represent what is known in geometry as a pair of conjugate semi-diameters of the ellipse.

The simplest way to determine the magnitudes of the semi-axes a and b from the initial conditions (that is, from $\mathbf{c}$ and $\mathbf{d}$) is to use the constancy of E and $\mathbf{J}$. Equating the expressions for these quantities, (4.7) and (4.5), to the similar expressions in terms of $\mathbf{a}$ and $\mathbf{b}$, we obtain

$$a^2+b^2 = c^2+d^2,$$

$$\mathbf{a}\wedge\mathbf{b} = \mathbf{c}\wedge\mathbf{d}.$$

(These equations may also be obtained directly from (4.9).) If α is the angle between $\mathbf{c}$ and $\mathbf{d}$, the second equation yields

$$ab = cd\sin\alpha.$$

Thus we obtain for a^2 or b^2 the quadratic equation

$$a^4-(c^2+d^2)a^2+c^2d^2\sin^2\alpha = 0. \tag{4.11}$$

4.2 The Conservation Laws

We now consider the general case of a central conservative force. It corresponds, as we saw in §3.6, to a potential energy function $V(r)$ depending only on r. There are two conservation laws, one for energy,

$$\tfrac{1}{2}m\dot{\mathbf{r}}^2+V(r) = E = \text{constant},$$

and one for angular momentum

$$m\mathbf{r}\wedge\dot{\mathbf{r}} = \mathbf{J} = \text{constant}.$$

According to the discussion of §3.3, the second of these laws implies that the motion is confined to a plane, so that the problem is effectively a two-dimensional one. Introducing polar co-ordinates r, θ in this plane, we may write the two conservation laws in the form

$$\begin{aligned}\tfrac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)+V(r) &= E,\\ mr^2\dot{\theta} &= J.\end{aligned} \tag{4.12}$$

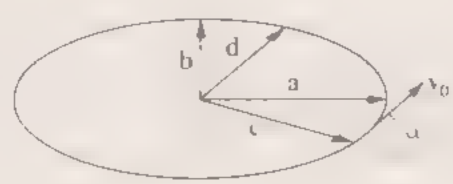


Fig. 4.1

A great deal of information can be obtained about the motion directly from these equations, without actually solving them to find r and θ as functions of the time. We note that $\dot{\theta}$ may be eliminated to yield an equation involving only r and $\dot{r}$,

$$\tfrac{1}{2}m\dot{r}^2+\frac{J^2}{2mr^2}+V(r) = E. \tag{4.13}$$

We shall call this the *radial energy equation.* For a given value of J, it has precisely the same form as the one-dimensional energy equation with a potential energy function

$$U(r) = \frac{J^2}{2mr^2}+V(r). \tag{4.14}$$

$$\frac{d}{dt}(m\dot{r}) = mr(\dot{\theta}^2+\sin^2\theta\,\dot{\phi}^2)-\frac{\partial V}{\partial r}$$

$$\frac{d}{dt}(mr^2\dot{\theta}) = mr^2\sin\theta\cos\theta\,\dot{\phi}^2-\frac{\partial V}{\partial \theta}$$

$$\frac{d}{dt}(mr^2\sin^2\theta\,\dot{\varphi}) = -\frac{\partial V}{\partial \varphi}$$

It is easy to understand the physical significance of the extra term $J^2\ 2mr^2$ in this 'effective potential energy'. It corresponds to a 'force' $J^2\ mr^3$. This is precisely the 'centrifugal force' $mr\dot{\theta}^2$ (the first term on the right side of the radial equation in (3.43)), expressed in terms of the constant J rather than the variable $\dot{\theta}$.

We can use the radial energy equation just as we did in §2.1. Since $\dot{r}^2$ is positive, the motion is limited to the range of values of r for which

$$U(r) = \frac{J^2}{2mr^2}+V(r)\leqslant E. \tag{4.15}$$

The maximum and minimum radial distances are given by the values of r for which the equality holds.

As an example, let us consider again the case of the isotropic oscillator, for which $V(r) = \frac{1}{2}kr^2$. The corresponding function $U(r)$ is shown in Fig. 4.2. It has a minimum (corresponding to a position of stable equilibrium in the one-dimensional case) at

$$r = \left(\frac{J^2}{mk}\right)^{1/4}. \tag{4.16}$$

When the value of E is equal to the minimum value of U, $\dot{r}$ must always be zero, and r is fixed at the position of the minimum. In this case, the particle must move in a circle around the origin. It is interesting to note that we could also obtain (4.16) by equating the attractive force kr to the centrifugal force in the circular orbit, J^2/mr^3.

For any larger value of E, the motion is confined to the region

$$b\leqslant r\leqslant a$$

between two limiting values of r, given by the solutions of the equation (4.15). If the particle is initially at a distance r_0 from the origin, and moving with velocity v_0 in a direction making an angle α with the radial direction (as in Fig. 4.1), then the values of E and J are

$$E = \tfrac{1}{2}mv_0^2 + \tfrac{1}{2}kr_0^2,$$

$$J = mr_0v_0 \sin \alpha.$$

Thus the equation for a or b becomes (on multiplying by $2r^2/k$)

$$r^4 - \left(r_0^2 + \frac{m}{k} v_0^2\right) r^2 + \frac{m}{k} r_0^2 v_0^2 \sin^2 \alpha = 0.$$

By (4.3), this is identical with the equation (4.11) found previously.

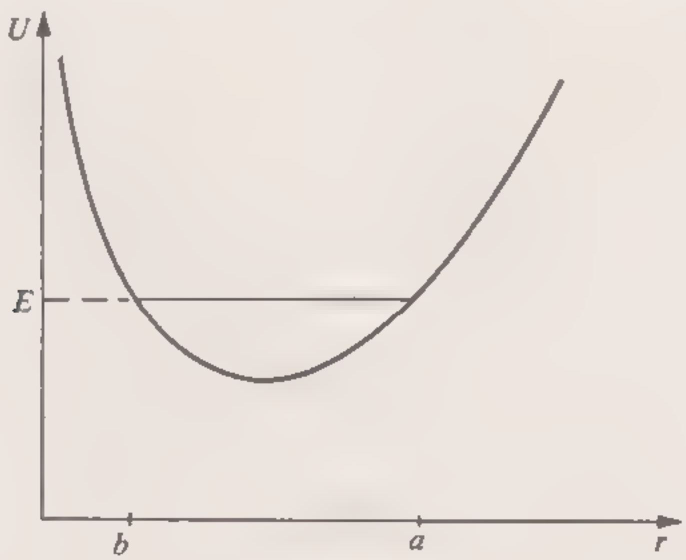


Fig. 4.2

4.3 The Inverse Square Law

We now consider a force

$$\mathbf{F} = \frac{k}{r^2}\,\hat{\mathbf{r}},$$

where k is a constant. The corresponding potential energy function is

$$V(r) = k/r.$$

The constant k may be either positive or negative; in the first case, the force is repulsive, and in the second, attractive.

The radial energy equation for this case is

$$\tfrac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + \frac{k}{r} = E. \tag{4.17}$$

It corresponds to the 'effective potential energy function'

$$U(r) = \frac{J^2}{2mr^2} + \frac{k}{r}.$$

Repulsive Case. We suppose first that $k>0$. Then $U(r)$ decreases monotonically from $+\infty$ at $r = 0$ to 0 at $r = \infty$. Thus, it has no minima, and circular motion is impossible, as is physically obvious.

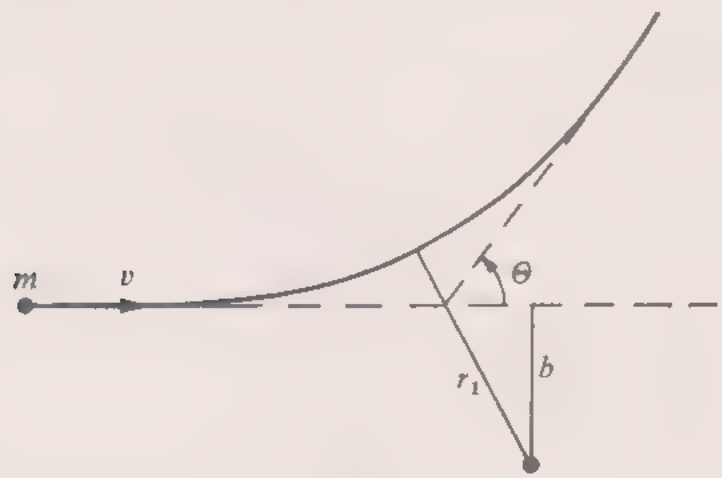


Fig. 4.3

For any positive value of E, there is a minimum value of r, r_1 say, which is the unique positive root of the equation $U(r) = E$, but no maximum value. If the radial velocity is initially inward, the particle must follow an orbit on which r decreases to r_1 (at which point the velocity is purely transverse), and then increases again without limit. As is well known, and as we shall show in the next section, the orbit is actually a hyperbola.

As an example, let us calculate the distance of closest approach for a charged particle of charge q moving in the field of a fixed point charge q'. We suppose that initially the particle is approaching the centre of force with velocity v (at a large distance) along a path which, if continued in a straight line, would pass the centre at a distance b. This distance b is known as the *impact parameter*, and will appear frequently in our future work. (See Fig. 4.3.) Since the

particle is initially at a great distance, its initial potential energy is negligible, and

$$E = \tfrac{1}{2}mv^2. \tag{4.18}$$

Moreover, since the component of $\mathbf{r}$ perpendicular to $\mathbf{v}$ is b, the angular momentum is

$$J = mbv. \tag{4.19}$$

The distance of closest approach r_1 is obtained by substituting these values, and $k = qq'$ (or in SI units, $k = qq'/4\pi\varepsilon_0$), in the radial energy equation, and setting $\dot{r} = 0$. This yields

$$r_1{}^2 - 2ar_1 - b^2 = 0, \quad a = qq'/mv^2. \tag{4.20}$$

The required solution is the positive root

$$r_1 = a + (a^2 + b^2)^{1/2}. \tag{4.21}$$

Attractive Case. We now suppose that $k<0$. It will be useful to define a quantity l, with the dimensions of length, by

$$l = \frac{J^2}{m|k|}. \tag{4.22}$$

Then the 'effective potential energy function' is

$$U(r) = |k|\left(\frac{l}{2r^2} - \frac{1}{r}\right).$$

Evidently, $U(\tfrac{1}{2}l) = 0$, and $U(r)$ has a minimum at $r = l$, with minimum value $U(l) = -|k|/2l$. This function is plotted in Fig. 4.4.

As for the one-dimensional potential well, different types of motion are possible according to the value of E. We may distinguish four cases:

(i) $E = -\dfrac{|k|}{2l}$.

This is the minimum value of U. Hence, $\dot{r}$ must always be zero, and the particle must move in a circle of radius l. Since the potential energy is $V = -|k|/l$, the kinetic energy $T = E - V$ is

$$T = \tfrac{1}{2}mv^2 = \frac{|k|}{2l}. \tag{4.23}$$

From this equation, we can deduce the orbital velocity v. (It may also be found by equating the attractive force $|k|/l^2$ to the 'centrifugal force' mv^2/l.) Note the interesting result that for a circular orbit the

potential energy is always twice as large in magnitude as the kinetic energy.

(ii) $-\frac{|k|}{2l} < E < 0$.

This case is illustrated by the lower line in Fig. 4.4. The radial distance is limited between a minimum distance r_1 and a maximum distance r_2, so the motion must be periodic. As we shall see, the orbit is in fact an ellipse.

(iii) $E = 0$.

In this case, there is a minimum distance $r_1 = \frac{1}{2}l$, but the maximum distance r_2 is infinite. Thus the particle has just enough energy to

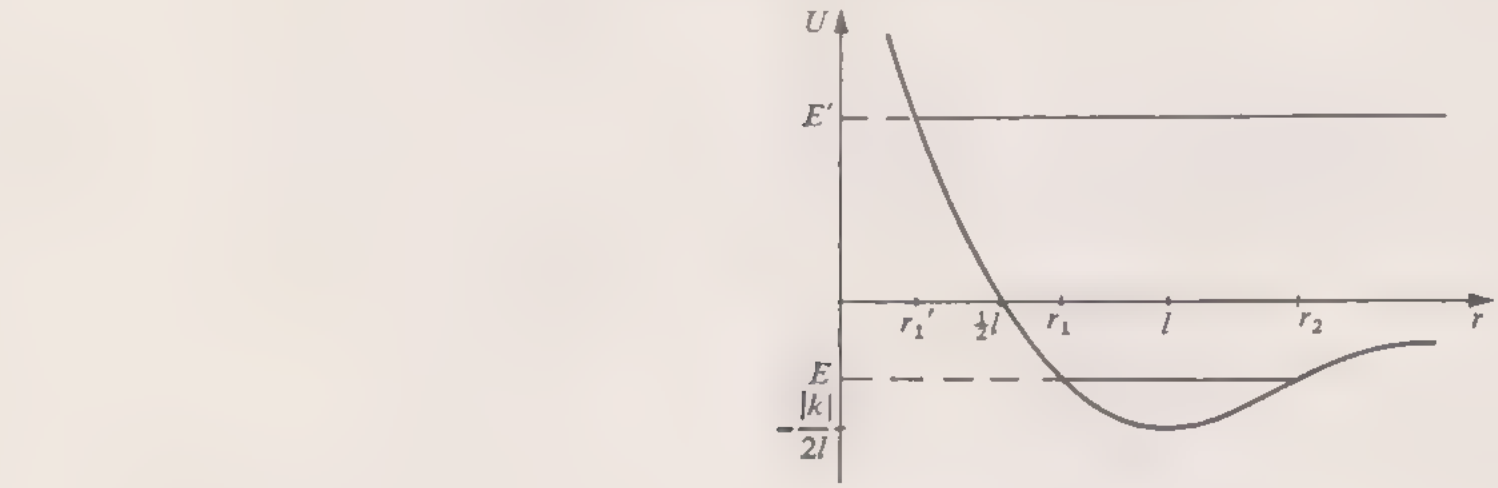


Fig. 4.4

escape to infinity, with kinetic energy tending to zero at large distances. The orbit will be shown to be a parabola.

(iv) $E > 0$.

This corresponds to the upper line in Fig. 4.4. Again, there is a minimum distance but no maximum distance. Now, however, the particle can escape to infinity with non-zero limiting velocity. The orbit is in fact a hyperbola.

Escape Velocity. As an example, we shall consider a projectile launched from the surface of the earth (which we take to be a sphere of mass M and radius R), with velocity v at an angle α to the vertical. The energy and angular momentum are

$$E = \tfrac{1}{2}mv^2 - \frac{GMm}{R}, \tag{4.24}$$

$$J = mRv \sin\alpha,$$

where G is the gravitational constant. To express the energy in terms of more familiar quantities, we note that the gravitational force on a particle at the earth's surface is

$$mg = \frac{GMm}{R^2},$$

from which we obtain the useful result

$$GM = R^2 g. \tag{4.25}$$

Thus,

$$E = \tfrac{1}{2}mv^2 - Rgm.$$

The projectile will escape to infinity provided that $E \geqslant 0$; or that its velocity v exceeds the *escape velocity*

$$v_e = (2Rg)^{1/2}.$$

Note that this condition is independent of the angle of projection α. Using the values $R = 6370$ km, $g = 980$ cm s^{-2}, we find for the escape velocity from the earth

$$v_e = 11{\cdot}2 \text{ km s}^{-1}.$$

If the projectile is launched with a velocity less than the escape velocity, it will reach some maximum height and then fall back. The maximum distance r is the larger root of the equation $U(r) = E$, or, with the values (4.24) for E and J,

$$(2Rg - v^2)r^2 - 2R^2gr + R^2v^2 \sin^2\alpha = 0.$$

For example, suppose that the launching velocity v is equal to the circular orbital velocity in an orbit just above the earth's surface, which, by (4.23) and (4.25), is

$$v_c = (Rg)^{1/2} = 7{\cdot}9 \text{ km s}^{-1}.$$

Then the equation reduces to

$$r^2 - 2Rr + R^2 \sin^2\alpha = 0,$$

so that

$$r = R(1 + \cos\alpha).$$

For vertical launching, the maximum distance is $2R$; for any other angle, it is less, and for almost horizontal launching, the orbit is nearly a circle of radius R.

This example illustrates the kind of problem which may readily be solved by using the radial energy equation. It is particularly

useful when we are interested only in r, and not in the polar angle θ or the time t.

Energy Levels of the Hydrogen Atom. This is not a problem which can be solved by the methods of classical mechanics alone. However, the energy levels obtained from quantum mechanics can also be obtained by imposing on the classical orbits an *ad hoc* 'quantization rule'. (Historically, this is how they were first obtained.) According to Bohr's 'old quantum theory', the electron in an atom cannot occupy any orbit, but only a certain discrete set of orbits. In the case of circular orbits, the quantization rule he imposed was that the angular momentum J should be an integral multiple of $\hbar$ (Planck's constant divided by 2π). The constant k in this case is $-e^2$, where e is the charge on the proton, or minus the charge on the electron. Thus, for $J = n\hbar$, the radius a_n of the orbit is, by (4.22),

$$a_n = \frac{J^2}{me^2} = n^2 a_1,$$

where a_1, the radius of the first Bohr orbit, is

$$a_1 = \frac{\hbar^2}{me^2} = 5{\cdot}3 \times 10^{-9}\ \text{cm}.$$

The corresponding energy levels are

$$E_n = -\frac{e^2}{2a_n} = -\frac{1}{n^2}\frac{e^2}{2a_1}.$$

These values agree well with the energies of atomic transitions, as determined from the spectrum of hydrogen.

4.4 Orbits

We now turn to the problem of determining the orbit of a particle moving under a central conservative force. This can be done by eliminating the time from the two conservation equations (4.12) to obtain an equation relating r and θ. The simplest way of doing this is to work not with r itself, but with the variable $u = 1/r$, and look for an equation determining u as a function of θ. Now

$$\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta},$$

whence

$$\dot{r} = \frac{dr}{d\theta}\dot{\theta} = -r^2\dot{\theta}\frac{du}{d\theta} = -\frac{J}{m}\frac{du}{d\theta}.$$

Thus, substituting for $\dot{r}$ in the radial energy equation (4.13), and multiplying by $2m/J^2$, we obtain

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 + u^2 = \frac{2m}{J^2}(E-V), \tag{4.26}$$

in which of course V is to be regarded as a function of $1/u$. This equation can be integrated (numerically, if not analytically) to give the equation of the orbit.

We shall consider explicitly only the case of the inverse square law, for which $V = ku$. We shall treat both cases $k>0$ and $k<0$ together. It will be useful, as in the preceding section, to define

$$l = \frac{J^2}{m|k|} = \pm\frac{J^2}{mk}, \tag{4.27}$$

where the upper sign refers to the repulsive case $k>0$, and the lower sign to the attractive case $k<0$. Then (4.26) becomes

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 + u^2 \pm \frac{2u}{l} = \frac{2E}{|k|l}.$$

To solve this equation, we multiply by l^2 and add 1 to both sides to complete the square. Then, if we introduce the new variable

$$z = lu \pm 1, \qquad \frac{\mathrm{d}z}{\mathrm{d}\theta} = l\frac{\mathrm{d}u}{\mathrm{d}\theta},$$

we may write the equation as

$$\left(\frac{\mathrm{d}z}{\mathrm{d}\theta}\right)^2 + z^2 = \frac{2El}{|k|} + 1 = e^2, \tag{4.28}$$

say. Note that since the left side of the equation is a sum of squares, it can have a solution only when the right side is also positive, in agreement with our earlier result that the minimum value of E is $-|k|/2l$. The general solution of this equation is

$$z = lu \pm 1 = e\cos(\theta-\theta_0),$$

where θ_0 is an arbitrary constant of integration. Thus, finally, we find that the orbit equation is, in the repulsive case,

$$r[e\cos(\theta-\theta_0)-1] = l, \tag{4.29}$$

and, in the attractive case,

$$r[e\cos(\theta-\theta_0)+1] = l. \tag{4.30}$$

These are the polar equations of conic sections, referred to a *focus* as origin. The constant e, the *eccentricity*, determines the shape of

the orbit; l determines its scale, and θ_0 its orientation relative to the co-ordinate axes. In the repulsive case, e must be greater than unity, and therefore $E>0$, since otherwise the square bracket in (4.29) is always negative. In the attractive case $e = 0$ when E has its minimum value $-|k|/2l$; the orbit is then the circle $r = l$. So long as $e<1$, or $E<0$, the square bracket in (4.30) is always positive, and there is a value of r for every value of θ. Thus the orbit is closed. When $e\geqslant 1$, or $E\geqslant 0$, however, r can become infinite when the square bracket vanishes. This is in agreement with our previous conclusion that the particle will escape to infinity if and only if its energy is positive.

Note that r takes its minimum value when $\theta = \theta_0$. Thus θ_0 specifies the direction of the point of closest approach. In the attractive case, the constant l also has a simple geometrical interpretation. It is the radial distance at right angles to this direction; that is, $r = l$ when $\theta = \theta_0 \pm \frac{1}{2}\pi$.

Elliptic Orbits ($E<0$, $e<1$). For most applications, it is best to use the orbit equation directly in its polar form. It is, however, a matter of simple algebra to rewrite it in the possibly more familiar Cartesian form. If we choose the axes so that $\theta_0 = 0$, we obtain

$$\frac{(x+ae)^2}{a^2}+\frac{y^2}{b^2} = 1,$$

where

$$a = \frac{l}{1-e^2} = \frac{|k|}{2|E|} \tag{4.31}$$

and

$$b^2 = al = \frac{J^2}{2m|E|}. \tag{4.32}$$

This is the equation of an ellipse with centre at $(-ae, 0)$ and semi-axes a and b. (See Fig. 4.5.) It is useful to note that the semi-major axis a is fixed by the value of the energy, while the semi-latus rectum l is fixed by the angular momentum.

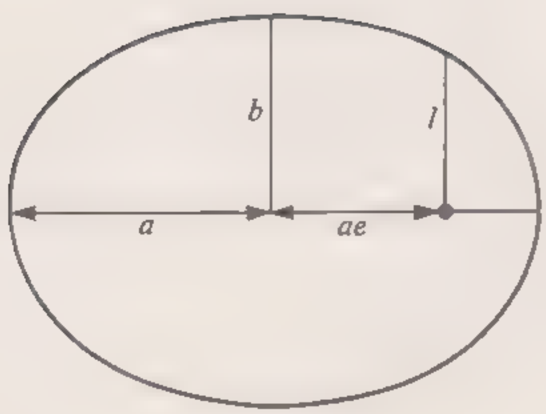


Fig. 4.5

The time taken by the particle to traverse any part of its orbit may be found from the relation (3.20) between angular momentum and rate of sweeping out area,

$$\frac{dA}{dt} = \frac{J}{2m}.$$

All we have to do is to evaluate the area swept out by the radius vector, and multiply by $2m/J$. In particular, since the area of the

ellipse is $A = \pi ab$, the orbital period is $\tau = 2m\pi ab/J$. Thus, by (4.27) and (4.32),

$$\left(\frac{\tau}{2\pi}\right)^2 = \frac{m}{|k|}a^3. \tag{4.33}$$

(It is easy to verify directly from (4.23) that this gives the correct period in a circular orbit of radius a.) For a planet or satellite orbiting round a central body of mass M,

$$\left(\frac{\tau}{2\pi}\right)^2 = \frac{a^3}{GM}. \tag{4.34}$$

This yields Kepler's third law of planetary motion: the square of the orbital period is proportional to the cube of the semi-major axis.

Hyperbolic Orbits ($E>0$, $e>1$). For both the attractive and repulsive cases, the Cartesian equation of the orbit is

$$\frac{(x-ae)^2}{a^2} - \frac{y^2}{b^2} = 1,$$

where

$$a = \frac{l}{e^2-1} = \frac{|k|}{2E}, \tag{4.35}$$

and

$$b^2 = al = \frac{J^2}{2mE}. \tag{4.36}$$

This is the equation of a hyperbola with centre at $(ae, 0)$ and semi-axes a and b. (See Fig. 4.6.) One branch of the hyperbola corresponds to the orbit in the attractive case and the other to that in the repulsive case. As before, a is determined by the energy, and l by the angular momentum. Note that by (4.18) and (4.19), the semi-minor axis b is identical with the impact parameter introduced earlier.

The directions in which r becomes infinite are, in the repulsive case, $\theta = \pm\cos^{-1}(1/e)$, and, in the attractive case, $\theta = \pi \pm \cos^{-1}(1/e)$. In both cases, therefore, the angle through which the particle is deflected from its original line of motion is

$$\Theta = \pi - 2\cos^{-1}(1/e).$$

This angle Θ is called the *scattering angle*. It will be useful to find the relation between this angle, the impact parameter b and the limiting velocity v. Since $E = \frac{1}{2}mv^2$, we have from (4.35), $a = |k|/mv^2$. But also, from (4.35) and (4.36),

$$b^2 = a^2(e^2-1) = a^2[\sec^2 \tfrac{1}{2}(\pi-\Theta)-1]$$
$$= a^2 \cot^2 \tfrac{1}{2}\Theta.$$

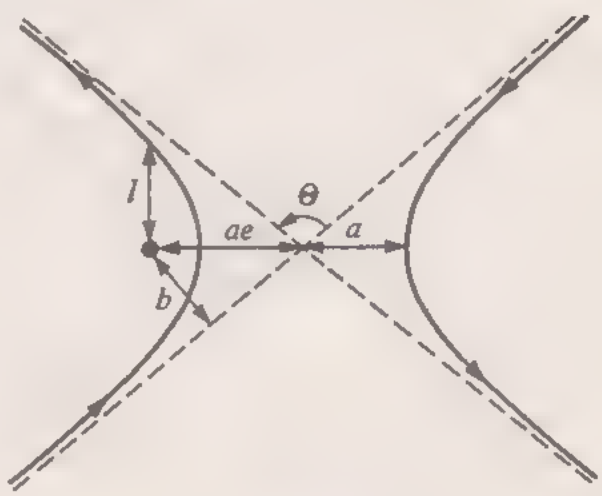


Fig. 4.6

Thus we obtain

$$b = \frac{|k|}{mv^2} \cot \tfrac{1}{2}\Theta. \tag{4.37}$$

4.5 Scattering Cross-Sections

One of the most important ways of obtaining information about the structure of small bodies is to bombard them with particles and measure the number of particles scattered in various directions. The angular distribution of scattered particles will depend on the shape of the target, and on the nature of the forces between the particles and the target. To be able to interpret the results of such an experiment, we must know how to calculate the expected angular distribution when the forces are given.

We shall consider first a particularly simple case. We suppose that the target is a fixed, hard (that is, perfectly elastic) sphere of radius R, and that a uniform, parallel beam of particles impinges on it. Let the particle flux in the beam, that is the number of particles crossing unit area normal to the beam direction per unit time, be f. Then the number of particles which strikes the target in unit time is

$$w = f\sigma, \tag{4.38}$$

where σ is the cross-sectional area presented by the target, namely

$$\sigma = \pi R^2. \tag{4.39}$$

Now let us consider one of these particles. We suppose that it impinges on the target with velocity v and impact parameter b. (See Fig. 4.7.) Then it will hit the target at an angle α to the normal given by

$$b = R \sin \alpha.$$

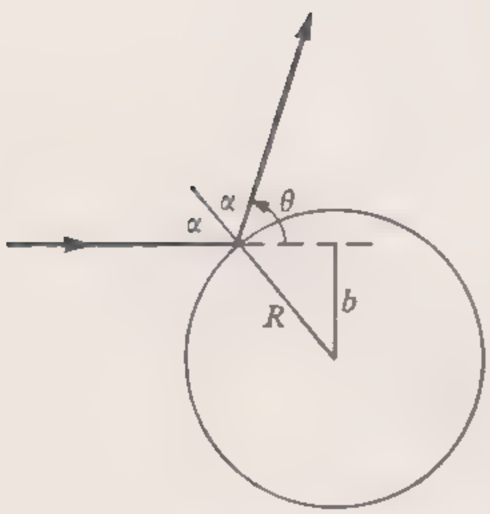


Fig. 4.7

The force on the particle is an impulsive central conservative force, corresponding to a potential energy function $V(r)$ which is zero for $r > R$, and rises very sharply in the neighbourhood of $r = R$. Thus the kinetic energy and angular momentum must be the same before and after the collision. We shall take the positive z direction ($\theta = 0$) to be the direction of motion of the incoming particles. Then the particle must move in a plane $\varphi =$ constant. From energy conservation, its velocity must be the same in magnitude before and after the collision. Then, from angular momentum conservation it follows that the particle will bounce off the sphere at an angle to the normal

equal to the incident angle α. Thus the particle is deflected through an angle $\theta = \pi - 2\alpha$, related to the impact parameter by

$$b = R \cos \tfrac{1}{2}\theta. \tag{4.40}$$

We can now calculate the number of particles scattered in a direction specified by the polar angles θ, φ, within an angular range $d\theta$, $d\varphi$. The particles scattered through angles between θ and $\theta + d\theta$ are

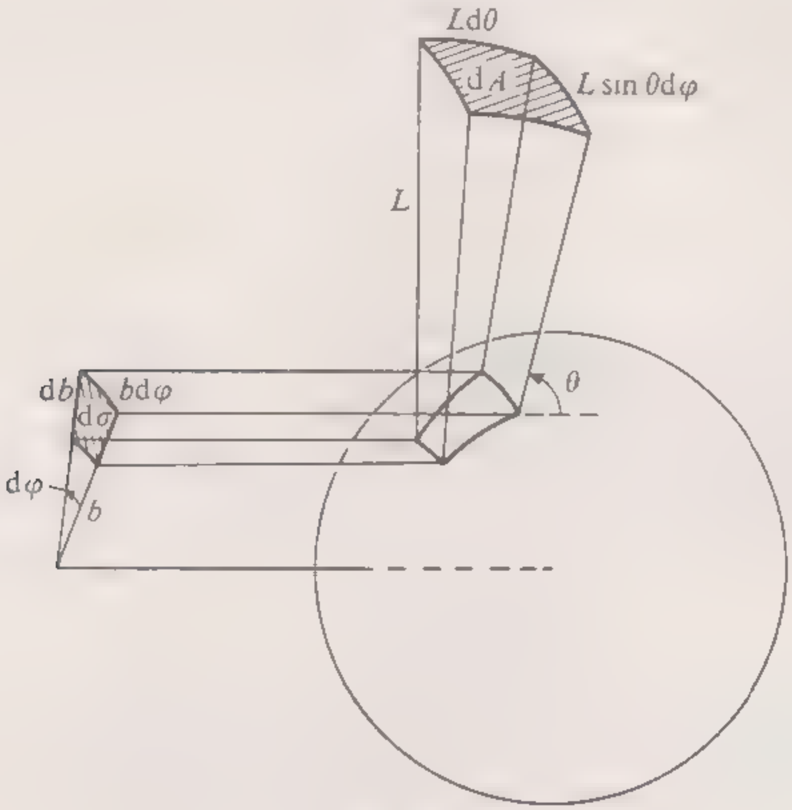


Fig. 4.8

those which came in with impact parameters between b and $b + db$, where

$$db = -\tfrac{1}{2}R \sin \tfrac{1}{2}\theta \, d\theta.$$

Consider now a cross-section of the incoming beam. The particles we are interested in are those which cross a small region of area

$$d\sigma = b|db| \, d\varphi. \tag{4.41}$$

(See Fig. 4.8.) Inserting the values of b and db, we find

$$d\sigma = \tfrac{1}{4}R^2 \sin \theta \, d\theta \, d\varphi. \tag{4.42}$$

The rate at which particles cross this area, and therefore the rate at which they emerge in the given angular range, is

$$dw = f \, d\sigma. \tag{4.43}$$

In order to measure this rate, we may place a small detector at a large distance from the target in the specified direction. We therefore want to express our result in terms of the cross-sectional area $\mathrm{d}A$ of the detector, and its distance L from the target (which we assume to be much greater than R). Now from the discussion of §3.4 (see the paragraph preceding (3.24)), we see that the element of area on a sphere of radius L is

$$\mathrm{d}A = L\,\mathrm{d}\theta \times L \sin\theta\,\mathrm{d}\varphi.$$

We define the *solid angle* subtended at the origin by the area $\mathrm{d}A$ to be

$$\mathrm{d}\Omega = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi,$$

so that

$$\mathrm{d}A = L^2\,\mathrm{d}\Omega. \tag{4.44}$$

The solid angle is measured in *steradians.* It plays the same role for a sphere as does the angle in radians for a circle; equation (4.44) is the analogue of the equation $\mathrm{d}s = L\,\mathrm{d}\theta$ for a circle of radius L. Just as the total angle subtended by an entire circle is 2π, so the total solid angle subtended by an entire sphere is

$$\iint \mathrm{d}\Omega = \frac{1}{L^2}\iint \mathrm{d}A = 4\pi.$$

The important quantity is not the cross-sectional area $\mathrm{d}\sigma$ itself but the ratio $\mathrm{d}\sigma\,\mathrm{d}\Omega$, which is called the *differential cross-section.* By (4.43) and (4.44), the rate $\mathrm{d}w$ at which particles enter the detector is

$$\mathrm{d}w = f\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\frac{\mathrm{d}A}{L^2}. \tag{4.45}$$

It is useful to note an alternative definition of the differential cross-section, which is applicable even when we cannot follow the trajectory of each individual particle, and therefore cannot say just which of the incoming particles are those that emerge in a given direction. We may define $\mathrm{d}\sigma/\mathrm{d}\Omega$ to be the ratio of the number of scattered particles per unit solid angle to the number of incoming particles per unit area. Then the rate at which particles are detected is obtained by multiplying the differential cross-section by the flux of incoming particles, and by the solid angle subtended at the target by the detector, as in (4.45). Note that $\mathrm{d}\sigma/\mathrm{d}\Omega$ has the dimensions of area (solid angle, like angle, is dimensionless); it is measured in square centimetres per steradian ($\mathrm{cm}^2\mathrm{sr}^{-1}$).

In the particular case of scattering from a hard sphere, the differential cross-section is, by (4.42),

$$\frac{d\sigma}{d\Omega} = \tfrac{1}{4}R^2. \qquad (4.46)$$

It has the special feature of being *isotropic*, or independent of the scattering angle. Thus the rate at which particles enter the detector is, in this case, independent of the direction in which it is placed. We note that the *total cross-section* (4.39) is correctly given by integrating (4.46) over all solid angles; in this case, we have merely to multiply by the total solid angle 4π.

4.6 Mean Free Path

The total cross-section σ is useful in discussing the attenuation of a beam of particles passing through matter. Let us consider first a particle moving through a medium containing n atoms per unit volume and suppose that the total cross-section for scattering by a single atom is σ. Consider a cylinder whose axis is the line of motion of the particle, and whose cross-sectional area is σ. Then the particle will collide with any atoms whose centres lie within the cylinder. Now the number of atoms in a length x of the cylinder is clearly $n\sigma x$. This is therefore the number of collisions made by the particle when it travels a distance x. Thus the mean distance travelled between collisions—the *mean free path* λ—is $x/n\sigma x$, or

$$\lambda = \frac{1}{n\sigma}. \qquad (4.47)$$

Now consider a beam of particles, with flux f, impinging normally on a wall. We wish to calculate the flux of particles which penetrate to a depth x without suffering a collision. Let this flux be $f(x)$, and consider a thin slice of the wall of thickness dx and area A at a depth x. The rate at which unscattered particles enter the slice is $Af(x)$, and the rate at which they emerge on the other side is $Af(x+dx)$. The difference between the two must be the rate at which collisions occur within the slice. Now the number of atoms in the slice is $nA\,dx$, and the total cross-sectional area presented by all of them is $\sigma nA\,dx$. Thus the rate at which collisions occur in the slice is $f(x)\sigma nA\,dx$. Equating the two quantities, we have

$$Af(x) - Af(x+dx) = f(x)\sigma nA\,dx,$$

or

$$\frac{df(x)}{dx} = -n\sigma f(x).$$

This equation may be integrated to give

$$f(x) = C\,\mathrm{e}^{-n\sigma x},$$

where C is an arbitrary constant. But $f(0) = f$, so finally

$$f(x) = f\,\mathrm{e}^{-n\sigma x} = f\,\mathrm{e}^{-x/\lambda}. \tag{4.48}$$

Thus the number of particles in the beam decreases by a factor of $1/\mathrm{e}$ in a distance of one mean free path.

Very often, when we wish to study the structure of a small object, like an atom, it would be quite impractical to use a target consisting only of a single atom. Instead, we have to use a target containing a large number N of atoms. If the thickness of the target is x, the rate at which collisions occur within the target will be

$$Af(0) - Af(x) = A(1 - \mathrm{e}^{-n\sigma x})f.$$

If x is small compared to the mean free path, then we may retain only the linear term in the expansion of the exponential. In this case the rate is approximately $An\sigma xf$. Since $N = nAx$, this shows that for a thin target the number of particles scattered will be just N times the number scattered by a single atom. For a thick target, it would be less, because the atoms would effectively screen each other.

When the target is thin, the probability of multiple collisions, in which a particle strikes several target atoms in succession, will be small. If we assume that it is negligible, then we can conclude that the angular distribution of the scattered particles will also be the same as that from a single target atom. This will be the case if all the dimensions of the target are small in comparison to the mean free path. If we use a detector whose distance from the target is large in comparison to the target size (so that the scattering angle does not depend appreciably on which of the atoms of the target was struck), then the rate at which particles enter the detector will be given by

$$\mathrm{d}w = Nf\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\frac{\mathrm{d}A}{L^2}, \tag{4.49}$$

that is, just N times the rate for a single target atom.

4.7 Rutherford Scattering

We discuss in this section a problem which was of crucial importance in obtaining an understanding of the structure of the atom. In a classic experiment, performed in 1911, Rutherford bombarded atoms with α-particles (helium nuclei). Because these particles are much heavier than electrons, they are deflected only very slightly by

α-p 14°5
§7·3 (7·23) & (7·24) α-e⁻ 0°03

the electrons in the atom (see Chapter 7), and can therefore be used to study the heavy atomic nucleus. From observations of the angular distribution of the scattered α-particles, Rutherford was able to show that the law of force between α-particle and nucleus is the inverse square law down to very small distances. Thus he concluded that the positive charge is concentrated in a very small nuclear volume rather than being spread out over the volume of the atom.

We shall now calculate the differential cross-section for the scattering of a particle of charge q and mass m by a fixed point charge q'. The impact parameter b is related to the scattering angle θ by (4.37).*

$$b = \frac{|qq'|}{mv^2}\cot\tfrac{1}{2}\theta.$$

Thus,

$$\mathrm{d}b = \frac{|qq'|}{mv^2}\frac{-\tfrac{1}{2}\,\mathrm{d}\theta}{\sin^2\tfrac{1}{2}\theta},$$

so that, substituting in (4.41), we obtain

$$\mathrm{d}\sigma = \left(\frac{qq'}{mv^2}\right)^2\frac{\cos\tfrac{1}{2}\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi}{2\sin^3\tfrac{1}{2}\theta}.$$

Dividing by the solid angle, we obtain the differential cross-section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{a^2}{4\sin^4\tfrac{1}{2}\theta}, \qquad a = \frac{|qq'|}{mv^2}. \tag{4.50}$$

This is the *Rutherford scattering cross-section.*

We note that, in contrast to the differential cross-section for scattering by a hard sphere, this cross-section is strongly dependent both on the velocity of the incoming particle and on the scattering angle. It also increases rapidly with increasing charge. For scattering of an α-particle on a nucleus of atomic number Z, $a = 2Ze^2/mv^2$, where e is the electronic charge. Thus we expect the number of particles scattered to increase like Z^2 with increasing atomic number.

We saw in §4.3 that the minimum distance of approach is given by (4.21). Thus to investigate the structure of the atom at small distances, we must use high-velocity particles, for which a is small, and examine the large-angle scattering, corresponding to particles with small impact parameter. The cross-section (4.50) is large for small values of the scattering angle, but physically it is the large-angle scattering which is of interest. For the fact that particles can be scattered through large angles is an indication that there are very

* In SI units, qq' should be replaced by $qq'/4\pi\varepsilon_0$.

§6.3 Spherical charge distribution

strong forces acting at short distances. If the positive nuclear charge were spread out over a large volume, the force would be the inverse-square-law force only down to a distance equal to the radius of the charge distribution. Beyond this point, it would decrease as we go to even smaller distances. (See Chapter 6.) Consequently, the particles which penetrate to within this distance would experience a weaker force than the inverse square law predicts, and would be scattered through smaller angles.

A peculiar feature of the differential cross-section (4.50) is that the corresponding *total* cross-section is infinite. This is a consequence of the infinite range of the Coulomb force. However far away from the nucleus a particle may be, it still experiences some force, and is scattered through a non-zero (though small) angle. Thus the total number of particles scattered through any angle, however small, is indeed infinite. We can easily calculate the number of particles scattered through any angle greater than some lower limit θ_0. These are the particles which had impact parameters b less than $b_0 = a \cot \frac{1}{2}\theta_0$. The corresponding cross-section is therefore

$$\sigma(\theta > \theta_0) = \pi b_0^2 = \pi a^2 \cot^2 \tfrac{1}{2}\theta. \tag{4.51}$$

(This may also be obtained by integration of the differential cross-section (4.50).)

4.8 Summary

For a particle moving under any central conservative force, information about the radial motion may be obtained from the radial energy equation, which results from eliminating $\dot{\theta}$ between the conservation equations for energy and angular momentum. The values of E and J can be determined from the initial conditions, and this equation then tells us the radial velocity at any value of r.

When information about the angle θ is needed, then we must use the equation of the orbit. For the inverse square law, the orbit is an ellipse or hyperbola according as $E<0$ or $E>0$. The semi-major axis is fixed by E, and the semi-latus rectum by J.

If we are also concerned with finding the time taken to traverse part of the orbit, we can use the relation between the angular momentum and the rate of sweeping out area.

PROBLEMS

1 The orbits of the synchronous communications satellites have been chosen so that viewed from the earth they appear to be stationary. Find the radius of the orbits.

2 The semi-major axis of the orbit of Jupiter is 5·2 times that of the earth. Find its orbital period in years.

3 The moon's mass and radius are 0·012 M_E and 0·27 R_E (E = earth). For Jupiter the corresponding figures are 318 M_E and 11 R_E. Find in each case the gravitational acceleration at the surface, and the escape velocity.

4 The minimum distance of a comet from the sun is observed to be half the radius of the earth's orbit (assumed circular), and its velocity at that point is twice the orbital velocity of the earth. Find its velocity when it crosses the earth's orbit, and the angle at which the orbits cross. Will the comet subsequently escape from the solar system?

5 Show that the comet of Problem 4 crosses the earth's orbit at opposite ends of a diameter. Find the time it spends inside the earth's orbit. (To evaluate the area required, write the equation of the orbit in Cartesian co-ordinates, and compute $\int x\, dy$.)

6 A star of mass M and radius R is moving with velocity v through a cloud of particles of density ρ. If all the particles which collide with the star are trapped by it, show that the mass of the star will increase at a rate

$$\frac{dM}{dt} = \pi\rho v\left(R^2+\frac{2GMR}{v^2}\right).$$

Evaluate the fractional increase in mass per year if

$$M = 10^{28}\ \text{t}, \qquad R = 10^8\ \text{km}, \qquad v = 10\ \text{km/s},$$

and

$$\rho = 10^{-18}\ \text{g cm}^{-3}.$$

7 Find the polar equation of the orbit of an isotropic harmonic oscillator by solving the differential equation (4.26), and verify that it is an ellipse with centre at the origin. Check also that the period is given correctly by $\tau = 2mA/J$.

8 The eccentricity of the earth's orbit is $e = 0{\cdot}0167$. If the orbit is divided in two by the minor axis, show that the times spent in the two halves of the orbit are $\left(\frac{1}{2}\pm\frac{e}{\pi}\right)$ years, and evaluate the difference in days.

9 The radius of the orbit of Venus is 0·72 of that of the earth. (The orbits are assumed circular and coplanar.) A spaceship is to travel from the earth to Venus along an elliptical orbit which just touches each of the planetary orbits. Find the relative velocity of the spaceship with respect to the earth just after launching, and that relative to Venus just before landing, neglecting in each case the gravitational attraction of the planet. (Orbital velocity of earth = 30 km/s.)

10 Find the time taken for the trip described in the preceding question. (Note that this is half the orbital period in the spaceship orbit.) Where must Venus be in its orbit relative to the earth at the time of launching to ensure that it will be in the right place when the spaceship arrives?

11 Discuss the possible types of orbit under an inverse cube law force described by the potential energy function $V(r) = k/2r^2$. For the repulsive case ($k > 0$), find the differential cross-section for a particle of mass m approaching with velocity v.

12 A ballistic rocket is fired from the surface of the earth with velocity $v < (Rg)^{1/2}$ at an angle α to the vertical. Find the equation of its orbit, and

show that to achieve maximum range we should choose α so that $l = 2a - R$. Deduce that the maximum range is $2R\theta$ where

$$\sin\theta = \frac{v^2}{2Rg - v^2}.$$

If the maximum range is 3600 nautical miles, find the launching velocity, and the angle at which the rocket should be launched. (1 nautical mile = 1 minute of arc; $R = 6400$ km.)

p101

13 The potential energy of a particle of mass m is $V(r) = k/r + c/3r^3$, where $k < 0$, and c is a small constant. (The gravitational potential energy in the equatorial plane of the earth has approximately this form, because of its flattened shape—see Chapter 6.) Find the angular velocity ω in a circular orbit of radius a, and the angular frequency ω' of small radial oscillations about this circular orbit. Hence show that a nearly circular orbit is approximately an ellipse whose axes precess at an angular velocity $\Omega = (c/ka^2)\omega$.

14 A beam of particles strikes a wall containing 2×10^{23} atoms/cm³. Each atom behaves like a sphere of radius 3×10^{-10} cm. Find the thickness of wall which exactly half the particles will penetrate without scattering.

15 An α-particle of energy 2×10^{-8} erg is scattered by an aluminium atom through an angle of 90°. Calculate the distance of closest approach to the nucleus. (Atomic number of α-particle = 2; atomic number of Al = 13; electronic charge = $4{\cdot}8 \times 10^{-10}$ e.s.u.)

A beam of such particles with a flux of 3×10^4 cm^{-2} s^{-1} strikes a target containing 50 mg of aluminium. A detector of cross-sectional area 4 cm² is placed 60 cm from the target at right angles to the beam direction. Find the rate of detection of α-particles. (Atomic mass of Al = 27 amu; 1 amu = $1{\cdot}66 \times 10^{-24}$ g.)

Rotating Frames

Chapter 5

Hitherto, we have always used inertial frames, in which the laws of motion take on the simple form expressed in Newton's laws. There are, however, a number of problems which can most easily be solved by using a non-inertial frame. For example, when discussing the motion of a particle near the earth's surface, it is often convenient to use a frame which is rigidly fixed to the earth, and rotates with it. In this chapter, we shall find the equations of motion with respect to such a frame, and discuss some applications of them.

§1.1.C last line

5.1 Angular Velocity; Rate of Change of a Vector

Let us consider a solid body which is rotating with constant angular velocity ω about a fixed axis. Let $\mathbf{n}$ be a unit vector along the axis, whose direction is defined by the right-hand rule: it is the direction in which a right-hand-thread screw would move when turned in the direction of the rotation. Then we define the *vector angular velocity* $\boldsymbol{\omega}$ to be a vector of magnitude ω in the direction of $\mathbf{n}$, $\boldsymbol{\omega} = \omega\mathbf{n}$. Clearly angular velocity, like angular momentum, is an *axial* vector. (See §3.2.)

Last paragraph and a reference to Appendix A §A·2 ∴ axial = antisymmetric tensor of rank 2 based on 3 component vectors.

For example, for the earth, $\boldsymbol{\omega}$ is a vector pointing along the polar axis towards the north pole. Its magnitude is equal to 2π divided by the length of the *sidereal* day (the rotation period with respect to the fixed stars, which is less than that with respect to the sun by one part in 365), that is

approximately for the completion period of the orbit.

$$\omega = \frac{2\pi}{86\,164}\,\mathrm{s}^{-1} = 7{\cdot}29\times10^{-5}\,\mathrm{s}^{-1}. \tag{5.1}$$

If we take the origin to lie on the axis of rotation, then the velocity of a point of the body at position $\mathbf{r}$ is given by the simple formula

$$\mathbf{v} = \boldsymbol{\omega}\wedge\mathbf{r}. \tag{5.2}$$

To prove this, we note that the point moves with angular velocity ω round a circle of radius $\rho = r\sin\theta$. (See Fig. 5.1.) Thus its speed is

$$v = \omega\rho = \omega r\sin\theta = |\boldsymbol{\omega}\wedge\mathbf{r}|.$$

Moreover, the direction of $\mathbf{v}$ is that of $\boldsymbol{\omega}\wedge\mathbf{r}$; for clearly, $\mathbf{v}$ is perpendicular to the plane containing $\boldsymbol{\omega}$ and $\mathbf{r}$, and it is easy to see from the figure that its sense is correctly given by the right-hand rule. Thus (5.2) is correct both as regards magnitude and direction.

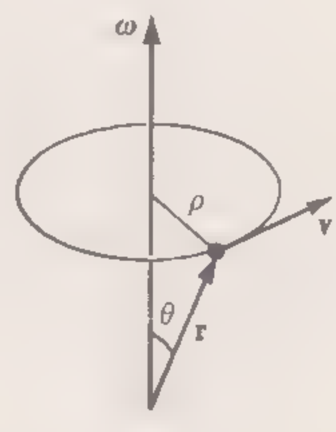


Fig. 5.1

It is not necessary that $\mathbf{r}$ should be the position vector of a point of the rotating body. If $\mathbf{a}$ is any vector fixed in the rotating body, then by the same argument

$$\frac{d\mathbf{a}}{dt} = \boldsymbol{\omega} \wedge \mathbf{a}. \tag{5.3}$$

In particular, if $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are unit vectors fixed in the body, then

$$\frac{d\mathbf{i}}{dt} = \boldsymbol{\omega} \wedge \mathbf{i}, \qquad \frac{d\mathbf{j}}{dt} = \boldsymbol{\omega} \wedge \mathbf{j}, \qquad \frac{d\mathbf{k}}{dt} = \boldsymbol{\omega} \wedge \mathbf{k}. \tag{5.4}$$

For example, if $\boldsymbol{\omega}$ is in the $\mathbf{k}$ direction, then

$$\frac{d\mathbf{i}}{dt} = \omega\mathbf{j}, \qquad \frac{d\mathbf{j}}{dt} = -\omega\mathbf{i}, \qquad \frac{d\mathbf{k}}{dt} = 0.$$

(Compare Fig. 5.2.)

Now consider a vector $\mathbf{a}$ specified with respect to the rotating axes $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, by the components a_x, a_y, a_z, so that

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}.$$

(Our discussion applies quite generally, but it may be helpful to think of $\mathbf{a}$ as the position vector of a particle with respect to the earth.) We must distinguish two kinds of 'rates of change', and it will be convenient to suspend for the moment the convention whereby $d\mathbf{a}/dt$ and $\dot{\mathbf{a}}$ mean the same thing. We shall denote by $d\mathbf{a}/dt$ the rate of change of $\mathbf{a}$ as measured by an inertial observer at rest relative to the origin, and by $\dot{\mathbf{a}}$ the rate of change as measured by an observer rotating with the solid body. We wish to find the relation between these two rates of change.

Now, although our two observers differ about the rate of change of a vector, they will always agree about the rate of change of any scalar quantity, and in particular about the rates of change of the components a_x, a_y, a_z. (Note that for the inertial observer, these are *not* the components of $\mathbf{a}$ with respect to his own set of axes. An observer outside the earth will always agree with one on the earth about the latitude and longitude of a particle, though these are not the co-ordinates he would naturally use himself to describe its position.) Hence we can write

$$\frac{da_x}{dt} = \dot{a}_x, \qquad \frac{da_y}{dt} = \dot{a}_y, \qquad \frac{da_z}{dt} = \dot{a}_z. \tag{5.5}$$

According to the observer on the rotating body, the rate of change of $\mathbf{a}$ is fully described by the rates of change of its components, so that

$$\dot{\mathbf{a}} = \dot{a}_x\mathbf{i} + \dot{a}_y\mathbf{j} + \dot{a}_z\mathbf{k}. \tag{5.6}$$

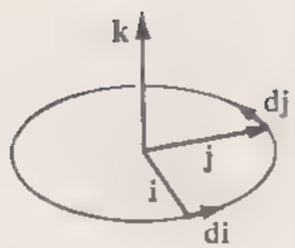


Fig. 5.2

However, to the inertial observer, the axes **i**, **j**, **k** are themselves changing with time, so that

$$\frac{d\mathbf{a}}{dt} = \left(\frac{da_x}{dt}\mathbf{i}+\frac{da_y}{dt}\mathbf{j}+\frac{da_z}{dt}\mathbf{k}\right)+\left(a_x\frac{d\mathbf{i}}{dt}+a_y\frac{d\mathbf{j}}{dt}+a_z\frac{d\mathbf{k}}{dt}\right)$$
$$= (\dot{a}_x\mathbf{i}+\dot{a}_y\mathbf{j}+\dot{a}_z\mathbf{k})+\boldsymbol{\omega}\wedge(a_x\mathbf{i}+a_y\mathbf{j}+a_z\mathbf{k}),$$

by (5.5) and (5.4). Thus, finally, we obtain

$$\frac{d\mathbf{a}}{dt} = \dot{\mathbf{a}}+\boldsymbol{\omega}\wedge\mathbf{a}. \tag{5.7}$$

Applied to a position vector **r**, this result is almost obvious, for it states that the velocity with respect to an inertial observer is the sum of the velocity $\dot{\mathbf{r}}$ with respect to the rotating frame, and the velocity $\boldsymbol{\omega}\wedge\mathbf{r}$ of a particle at r rotating with the body. We have given a detailed proof because of the central importance of this result.

We shall frequently encounter in our later work equations similar to (5.3). It is important to realize that one can reverse the argument which led up to it. If **a** satisfies this equation, then it must be a vector of constant length rotating with angular velocity ω about the direction of $\boldsymbol{\omega}$. For, if we introduce a frame of reference rotating with angular velocity $\boldsymbol{\omega}$, then according to (5.7), $\dot{\mathbf{a}} = \mathbf{0}$, so that **a** is fixed in the rotating frame.

5.2 Particle in a Uniform Magnetic Field

A particle of charge q moving with velocity **v** in a magnetic field **B**(**r**) experiences a force proportional to its velocity, and perpendicular to it,

$$\mathbf{F} = \frac{q}{c}\mathbf{v}\wedge\mathbf{B}, \tag{5.8}$$

where c is the velocity of light. (See Appendix B. In this formula charge is measured in *electrostatic* units. In SI or electromagnetic units, the factor of c should be deleted here and below.) The equation of motion is then

$$m\frac{d\mathbf{v}}{dt} = \frac{q}{c}\mathbf{v}\wedge\mathbf{B}. \tag{5.9}$$

Now let us suppose that the magnetic field is uniform (independent of position) and constant in time. Then (5.9) has precisely the form of (5.3),

$$\frac{d\mathbf{v}}{dt} = \boldsymbol{\omega}\wedge\mathbf{v},$$

where

$$\boldsymbol{\omega} = -\frac{q}{mc}\mathbf{B}. \tag{5.10}$$

Hence, by the argument of the preceding section, the velocity vector $\mathbf{v}$ rotates about the direction of $\mathbf{B}$ with constant angular velocity. It follows that the path of the particle must be a helix (spiral) whose axis points along the field lines. (See Fig. 5.3.) If we take the direction of $\mathbf{B}$ to be the z direction, then $\mathbf{v}$ has a constant z component, and a component in the xy-plane which rotates around a circle with angular velocity ω. In particular, if $\mathbf{v}$ is initially parallel to $\mathbf{B}$, then the particle will continue to move with uniform velocity along a field line. The other limiting case, in which $\mathbf{v}$ is perpendicular to $\mathbf{B}$, is more interesting. In that case, the particle describes a circle of radius r given by

$$r = \frac{v}{\omega} = \frac{mcv}{qB}. \tag{5.11}$$

Fig. 5.3

This effect is used in accelerating particles in the *cyclotron*. In order to prevent the particles from escaping, a strong magnetic field is employed to constrain them to move in circles. To accelerate them, an alternating voltage is applied between two D-shaped pole pieces enclosing the two halves of the circles. (See Fig. 5.4.) The angular frequency of this alternating voltage is chosen to coincide with the *cyclotron frequency*

$$\omega_c = \frac{qB}{mc}. \tag{5.12}$$

Thus, by the time the particles have made a half-revolution, the direction of the electric field has reversed, and they experience an accelerating field each time they cross the gap. Each semicircle is therefore slightly larger than the preceding one, and the particles spiral outwards with increasing energy. Finally, one obtains a beam of high-velocity particles at the outer edge of the cyclotron.

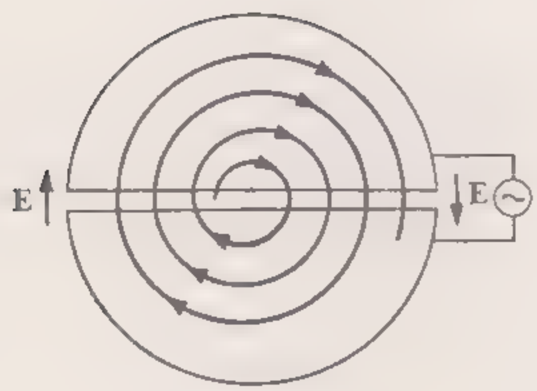


Fig. 5.4

This method can be used only so long as the velocity remains small compared to the velocity of light; for, according to the relativity theory, the angular frequency ω_c is not precisely constant when the velocity is large, but decreases with increasing velocity. Quite similar methods can be used, however, even at the highest energies.

5.3 Acceleration; Apparent Gravity

We can use the formula (5.7) twice to obtain the relation between the absolute acceleration $d^2\mathbf{r}\ dt^2$ of a particle and its acceleration $\ddot{\mathbf{r}}$

relative to a rotating frame. The velocity of the particle relative to an inertial observer is

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} + \boldsymbol{\omega}\wedge\mathbf{r}. \tag{5.13}$$

Applying the same formula to the rate of change of $\mathbf{v}$, we have

$$\frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} = \dot{\mathbf{v}} + \boldsymbol{\omega}\wedge\mathbf{v}. \tag{5.14}$$

But, from (5.13),

$$\dot{\mathbf{v}} = \ddot{\mathbf{r}} + \boldsymbol{\omega}\wedge\dot{\mathbf{r}},$$

and

$$\boldsymbol{\omega}\wedge\mathbf{v} = \boldsymbol{\omega}\wedge\dot{\mathbf{r}} + \boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{r}).$$

Hence, substituting in (5.14), we obtain

$$\frac{d^2\mathbf{r}}{dt^2} = \ddot{\mathbf{r}} + 2\boldsymbol{\omega}\wedge\dot{\mathbf{r}} + \boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{r}). \tag{5.15}$$

The second term on the right is called the *Coriolis* acceleration, and the third term the *centripetal* acceleration. The latter is directed inwards towards the axis, and perpendicular to it, as may be seen by writing it in the form

$$\boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{r}) = (\boldsymbol{\omega}\cdot\mathbf{r})\boldsymbol{\omega} - \omega^2\mathbf{r}.$$

The most important application of (5.15) is to a particle moving near the surface of the earth. For a particle moving under gravity, and under an additional mechanical force $\mathbf{F}$, the equation of motion is

$$m\frac{d^2\mathbf{r}}{dt^2} = m\mathbf{g} + \mathbf{F},$$

where $\mathbf{g}$ is a vector of magnitude g pointing downward. Using (5.15), and moving all the terms except the relative acceleration to the right side of the equation, we can write it as

$$m\ddot{\mathbf{r}} = m\mathbf{g} + \mathbf{F} - 2m\boldsymbol{\omega}\wedge\dot{\mathbf{r}} - m\boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{r}). \tag{5.16}$$

The last two terms on the right are *apparent* forces, which arise because of the non-inertial nature of the frame of reference. We shall postpone the discussion of the effects of the third term—the *Coriolis*

force—to the next section, and consider here the last term, which is the familiar *centrifugal force*.

This force is a slowly varying function of position, proportional, like the gravitational force, to the mass of the particle. When we make a laboratory measurement of the acceleration due to gravity, what we actually measure is not $\mathbf{g}$ but

$$\mathbf{g}^* = \mathbf{g} - \boldsymbol{\omega} \wedge (\boldsymbol{\omega} \wedge \mathbf{r}).$$

In particular, a plumb line does not point directly towards the earth's centre, but is swung outwards through a small angle by the centrifugal force. (See Fig. 5.5, in which the effect is greatly exaggerated.) Let us consider a point in colatitude ($\frac{1}{2}\pi$—latitude) θ. Then

$$|\boldsymbol{\omega} \wedge (\boldsymbol{\omega} \wedge \mathbf{r})| = \omega |\boldsymbol{\omega} \wedge \mathbf{r}| = \omega^2 r \sin \theta.$$

Thus the horizontal and vertical components of $\mathbf{g}^*$ are

$$g_h^* = \omega^2 r \sin \theta \cos \theta,$$

$$g_v^* = g - \omega^2 r \sin^2 \theta.$$

The magnitude of the centrifugal force may be found by inserting the value (5.1) for ω, and for r the mean radius of the earth, 6370 km. We find

$$\omega^2 r = 3{\cdot}4 \text{ cm s}^{-2}. \tag{5.17}$$

Since $\omega^2 r \ll g$, the angle α between the apparent and true verticals is approximately

$$\alpha \approx \frac{g_h^*}{g_v^*} \approx \frac{\omega^2 r}{g} \sin \theta \cos \theta.$$

The maximum value occurs at $\theta = 45°$, and is about 0°6′.

At the pole, there is no centrifugal force, and $g^* = g$. On the equator, $g^* = g - \omega^2 r$. Thus we might expect the measured value of the acceleration due to gravity to be larger at the pole by 3·4 cm s^{-2}. The actual measured difference is somewhat larger than this,

$$\Delta g^* = g^*_{\text{pole}} - g^*_{\text{eq.}} = 5{\cdot}2 \text{ cm s}^{-2}. \tag{5.18}$$

This discrepancy arises from the fact that the earth is not a perfect sphere, but more nearly spheroidal in shape, flattened at the poles. Thus the gravitational acceleration g, even excluding the centrifugal term, is itself larger at the pole than on the equator. These two effects are not really independent, for the flattening of the earth is a consequence of its rotation. We shall discuss this point in the next chapter.

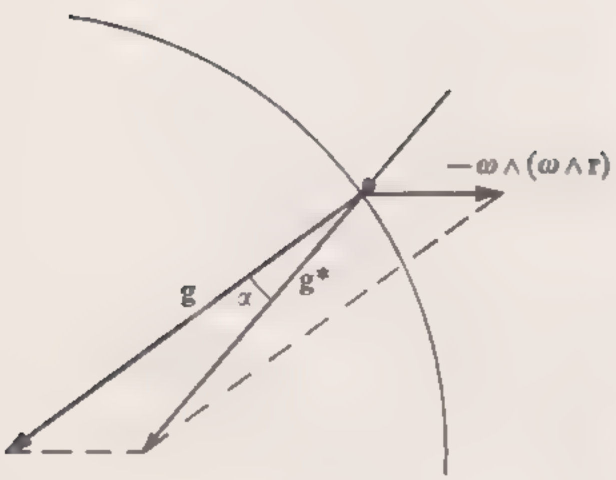


Fig. 5.5

Surface of a Rotating Liquid. As another example of the effect of the centrifugal force, we shall consider the problem of finding the surface of a liquid in a vessel rotating with angular velocity ω. We shall assume that the liquid is rotating with the same angular velocity, so that in the rotating frame it is at rest.

Now the centrifugal force is conservative, and corresponds to a potential energy

$$V_{\text{cent.}} = -\tfrac{1}{2}m\omega^2\rho^2 = -\tfrac{1}{2}m\omega^2(x^2+y^2). \tag{5.19}$$

Consider a particle on the surface of the liquid. Unless the surface is one on which the potential energy is constant, the particle will tend to move towards regions of lower potential energy. Hence if the liquid is in equilibrium under the gravitational and centrifugal forces, its surface must be an equipotential surface,

$$gz - \tfrac{1}{2}\omega^2(x^2+y^2) = \text{constant}.$$

This is a paraboloid of revolution about the z-axis.

5.4 Coriolis Force

The Coriolis force $-2m\boldsymbol{\omega}\wedge\dot{\mathbf{r}}$ is an apparent velocity-dependent force arising from the earth's rotation. To understand its physical origin, it may be helpful to consider a flat rotating disc. Suppose that a particle moves across the disc under no forces, so that an inertial observer sees it moving diametrically across in a straight line. (See Fig. 5.6(a).) Then, because the disc is rotating, an observer on the disc will see the particle crossing successive radii, following a curved track as in Fig. 5.6(b). If he is unaware that the disc is rotating, he will ascribe this curvature to a force acting on the particle at right angles to its velocity. This is the Coriolis force.

We may regard Fig. 5.6 as representing the earth viewed from the north pole. Thus we see that the effect of the Coriolis force is to make a particle deviate to the right in the northern hemisphere, and to the left in the southern hemisphere. (It also has a vertical component, in general.) It is responsible for some well-known meteorological phenomena, which we shall discuss below.

First, however, we shall examine some effects of the Coriolis force which are observable in the laboratory. Let us consider a particle moving near the surface of the earth in colatitude θ. We shall suppose that the distance through which it moves is sufficiently small for both the gravitational and centrifugal forces to be effectively

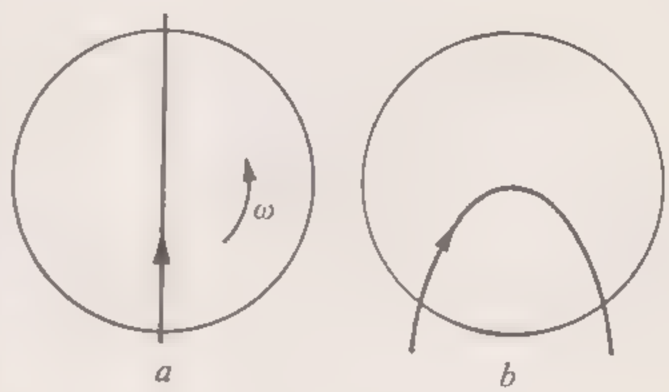


Fig. 5.6

constant. Then we may combine them in a constant effective gravitational acceleration $\mathbf{g}^*$. For convenience, we shall drop the star, and write this constant simply as $\mathbf{g}$. The equation of motion is then

$$m\ddot{\mathbf{r}} = m\mathbf{g}+\mathbf{F}-2m\boldsymbol{\omega}\wedge\dot{\mathbf{r}}. \tag{5.20}$$

Let us choose our axes so that $\mathbf{i}$ is east, $\mathbf{j}$ north and $\mathbf{k}$ up. (See Fig. 5.7.) The angular velocity vector then has the components

$$\boldsymbol{\omega} = (0,\ \omega\sin\theta,\ \omega\cos\theta).$$

Hence the Coriolis force is

$$-2m\boldsymbol{\omega}\wedge\dot{\mathbf{r}} = 2m\omega(\dot{y}\cos\theta-\dot{z}\sin\theta,\ -\dot{x}\cos\theta,\ \dot{x}\sin\theta). \tag{5.21}$$

Freely Falling Body. Let us suppose first that the particle is dropped from rest at a height h above the ground. Neglecting the Coriolis force, its motion is described by

$$x = 0, \qquad y = 0, \qquad z = h-\tfrac{1}{2}gt^2.$$

We shall calculate the effect of the Coriolis force to first order; that is, we neglect terms of order ω^2. Since the Coriolis force (5.21) contains a factor of ω, this means that we may substitute for $\dot{\mathbf{r}}$ the zero-order value

$$\dot{x} = 0, \qquad \dot{y} = 0, \qquad \dot{z} = -gt.$$

The equations of motion (5.20) then read

$$m\ddot{x} = 2m\omega gt\sin\theta, \qquad m\ddot{y} = 0, \qquad m\ddot{z} = -mg.$$

The solution, with appropriate initial conditions, is

$$x = \tfrac{1}{3}\omega gt^3\sin\theta, \qquad y = 0, \qquad z = h-\tfrac{1}{2}gt^2.$$

The particle will hit the ground, $z = 0$, at a point east of that vertically below its point of release, at a distance

$$x = \tfrac{1}{3}\omega\left(\frac{8h^3}{g}\right)^{1/2}\sin\theta. \tag{5.22}$$

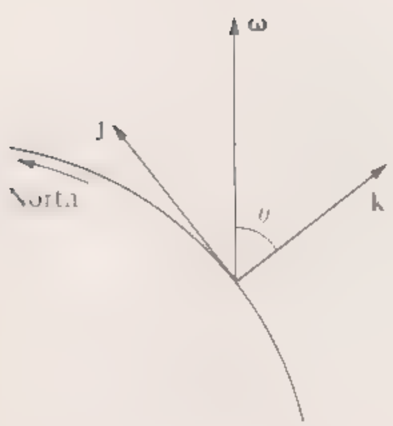


Fig. 5.7

For example, if a particle is dropped from a height of 100 m in latitude 45°, the deviation is about 1·6 cm.

It is instructive to consider how an inertial observer would describe this experiment. Since the particle is dropped from rest relative to the earth, it has a component of velocity to the east relative to the inertial observer. As it falls, its angular momentum about the earth's axis remains constant, and therefore its angular velocity increases, so that it gets ahead of the ground beneath it. Figures

5.8(a) and 5.8(b) show the experiment as it appears to an observer on the earth, and to an inertial observer, respectively.

Foucault's Pendulum. Another way of observing the effect of the Coriolis force is to use Foucault's pendulum. This is simply an ordinary pendulum free to swing in any direction, and carefully arranged to be perfectly symmetric, so that its periods of oscillation in all directions are precisely equal. (It must also be fairly heavy, so that it will go on swinging freely for several hours at least, despite the resistance of the air.) If the amplitude is small, the pendulum equation is simply the equation of two-dimensional simple harmonic motion. The vertical component of the Coriolis force is negligible, for it is merely a small correction to g, whose sign alternates on each half-period. (Even for a velocity $\dot{x}$ as large as 10 m/s, we have $2\,\omega\dot{x} = 0{\cdot}15$ cm s$^{-2} \ll g$.) The important components are the horizontal ones. For small amplitude, the velocity of the pendulum bob is almost horizontal, so that $\dot{z} \approx 0$. Thus the equations of motion for the x and y co-ordinates are

$$\begin{aligned} \ddot{x} &= -\frac{g}{l}\,x + 2\omega\dot{y}\cos\theta, \\ \ddot{y} &= -\frac{g}{l}\,y - 2\omega\dot{x}\cos\theta, \end{aligned} \tag{5.23}$$

or, in vector notation,

$$\ddot{\mathbf{r}} = -\frac{g}{l}\,\mathbf{r} - 2\omega\cos\theta\,\mathbf{k}\wedge\dot{\mathbf{r}}. \tag{5.24}$$

Let us first suppose that the pendulum is at the north pole ($\theta = 0$). Then it is clear from the equation of motion in a non-rotating frame that it must swing in a fixed direction in space, while the earth rotates beneath it. Thus, relative to the earth, its oscillation plane must rotate around the vertical with angular velocity $-\omega$. Now, at any other latitude, the only difference in (5.24) is that in place of the angular velocity ω we have only its vertical component $\omega\cos\theta\,\mathbf{k}$. Hence we should expect that the pendulum rotates with angular velocity $-\Omega = -\omega\cos\theta$ around the vertical. In effect, we may regard the earth's surface in colatitude θ as rotating about the vertical with angular velocity $\omega\cos\theta$ (and also with angular velocity $\omega\sin\theta$ about a horizontal north–south axis, but this component leads only to a vertical Coriolis force, which does not appreciably affect the motion).

We can verify this conclusion by obtaining an explicit solution of equations (5.23). A neat way of doing this is to combine the

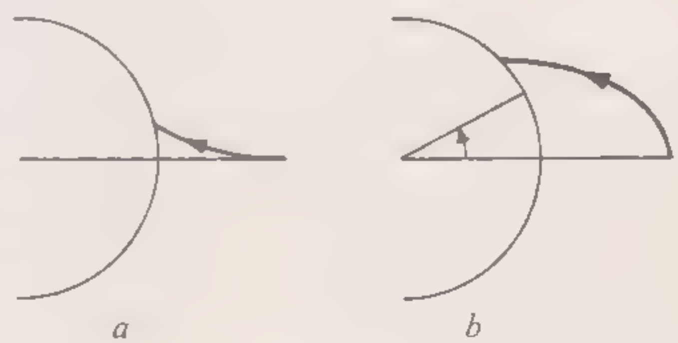


Fig. 5.8

two equations by writing $z = x+iy$. Then $\dot{y}-i\dot{x} = -i\dot{z}$, so that the equations become

$$\ddot{z}+2i\Omega\dot{z}+\omega_0^2 z = 0,$$

where $\Omega = \omega \cos\theta$ and $\omega_0^2 = g/l$. We look now for solutions of the form $z = Ae^{pt}$, and as in §2.5 find for p the equation

$$p^2+2i\Omega p+\omega_0^2 = 0.$$

The roots of this equation are $p = -i\Omega \pm i\omega_1$, where $\omega_1^2 = \omega_0^2+\Omega^2$. Hence the general solution of the equation for z is

$$z = Ae^{-i(\Omega-\omega_1)t}+Be^{-i(\Omega+\omega_1)t}.$$

In particular, if we set $A = B = \frac{1}{2}a$, we obtain a solution

$$z = ae^{-i\Omega t}\cos\omega_1 t,$$

or, in terms of x and y,

$$x = a\cos\Omega t\cos\omega_1 t, \qquad y = -a\sin\Omega t\cos\omega_1 t.$$

Initially, the oscillation is in the x direction. As time progresses, the amplitude $a\cos\Omega t$ of the x co-ordinate decreases, while that of the y co-ordinate, $-a\sin\Omega t$, grows. The solution represents an oscillation of amplitude a in a plane rotating with angular velocity $-\Omega$.

It should be noted that in deriving (5.23) we neglected terms of order Ω^2 (in particular, the variation in the centrifugal force is of this order). Thus the difference between the angular frequency ω_1 of our solution and ω_0 is not significant. The period of the pendulum will not be substantially affected.

At the pole, the plane of oscillation makes a complete revolution in just 24 hours. At any other latitude the period is greater than this, and is in fact $2\pi/\omega\cos\theta$. In latitude 45°, it is about 34 hours, while on the equator (where $\boldsymbol{\omega}$ is purely horizontal), it is infinite.

Cyclones and Trade Winds. We shall now consider some of the large-scale effects of the Coriolis force. Suppose that for some reason a region of low pressure arises in the northern hemisphere. The air around it will be pushed inwards by the force of the pressure gradient. As it starts to move, however, the Coriolis force causes it to curve to the right. Thus an anticlockwise rotation is set up around the low pressure zone. The process will continue until approximate equilibrium is established between the pressure acting inwards, and the Coriolis force (plus the centrifugal force of the rotation) acting outwards. This configuration is a cyclone, or depression, familiar to those who live in temperate latitudes. (As in the case of Foucault's

pendulum, there is no effect of this kind on the equator.) More generally, in these latitudes, the wind velocity is not directly from regions of high to low pressure, but more nearly along the isobars, keeping the low pressure on the left in the northern hemisphere.

The same effect is responsible, on a yet larger scale, for the direction of the trade winds. The heating of the earth's surface near the equator causes the air to rise, and be replaced by cooler air flowing in towards the equator. However, because of the Coriolis force, it does not flow directly north or south, but is made to deviate towards the west. Thus we have the north-east trade winds in the northern hemisphere, and the south-east trade winds in the southern hemisphere.

For reasons which are too complex to discuss in detail here, this pattern does not extent to high latitudes. Further from the equator, there are belts of high pressure, which, like the region around the equator itself, are characterized by light and variable winds. Beyond these, the direction of the pressure gradient is reversed. Thus around 40° to 50° N or S the prevailing wind direction is westerly rather than easterly. Near the poles, there is a further reversal, and the circumpolar winds blow from the east.

5.5 Larmor Effect

As a rather different example of the use of rotating frames, we shall consider the effect of a magnetic field on a particle of charge q moving in an orbit round a fixed point charge $-q'$. The equation of motion, including the magnetic force (5.8), is

$$m\frac{d^2\mathbf{r}}{dt^2} = -\frac{qq'}{r^2}\hat{\mathbf{r}} + \frac{q}{c}\frac{d\mathbf{r}}{dt}\wedge\mathbf{B}.$$

Let us rewrite this equation in terms of a rotating frame of reference, using (5.7) and (5.15). We obtain

$$\ddot{\mathbf{r}} + 2\boldsymbol{\omega}\wedge\dot{\mathbf{r}} + \boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{r}) = -\frac{qq'}{mr^2}\hat{\mathbf{r}} + \frac{q}{mc}(\dot{\mathbf{r}} + \boldsymbol{\omega}\wedge\mathbf{r})\wedge\mathbf{B}.$$

Now, if we choose

$$\boldsymbol{\omega} = -\frac{q}{2mc}\mathbf{B},$$

then the terms in $\dot{\mathbf{r}}$ drop out. The last term on the left only cancels half the last term on the right, however, so we obtain

$$\ddot{\mathbf{r}} = -\frac{qq'}{mr^2}\hat{\mathbf{r}} + \left(\frac{q}{2mc}\right)^2\mathbf{B}\wedge(\mathbf{B}\wedge\mathbf{r}).$$

We now assume that the magnetic field is sufficiently weak for the quadratic term in **B** to be negligible in comparison to the electrostatic force term. The necessary condition for this is that

$$\left(\frac{qB}{2mc}\right)^2 = \omega^2 \ll \frac{qq'}{mr^3} \approx \omega_0{}^2 \qquad (5.25)$$

where we have used (4.33) to express the right side in terms of the mean angular velocity ω_0 of the particle in its orbit.

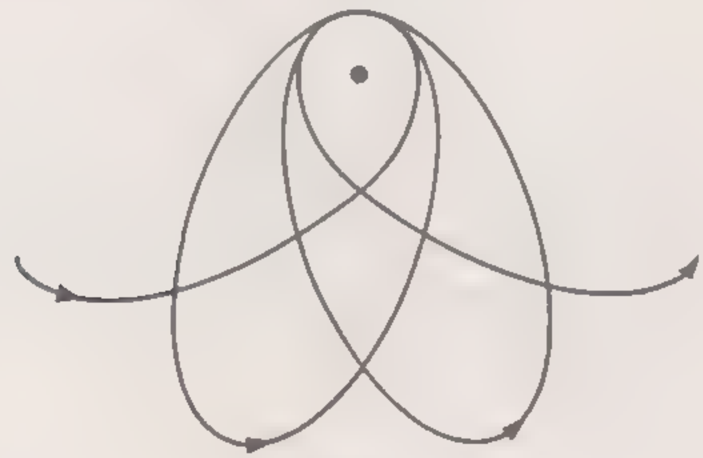


Fig. 5.9

When this condition is satisfied, we obtain the approximate equation

$$m\ddot{\mathbf{r}} = -\frac{qq'}{r^2}\,\hat{\mathbf{r}}.$$

But this is just the usual equation of motion for a particle in an inverse square law field. Consequently, the orbit in this rotating frame is an ellipse. In the original non-rotating frame it is a slowly precessing ellipse, precessing with angular velocity ω. (See Fig. 5.9, which is drawn for the special case in which the orbit lies in the plane normal to **B**. In general, it will be inclined, and the plane of the orbit will precess about the direction of **B**.) Note that, in view of the restriction (5.25), the axes of the ellipse will precess only through a small angle on each revolution.

This effect is known as the *Larmor effect*, and the angular velocity of precession,

$$\omega_{\mathrm{L}} = \frac{qB}{2mc},$$

is called the *Larmor frequency*. Note that it is just half the cyclotron frequency. This difference arises from the factor of 2 in the Coriolis acceleration term of (5.15).

This effect leads to observable changes in the spectra emitted by atoms in the presence of a magnetic field, since the Bohr energy level corresponding to a given angular momentum is slightly shifted in the presence of a magnetic field. (The shift in the spectral lines is known as the Zeeman effect.)

5.6 Angular Momentum and the Larmor Effect

The Larmor effect is a particular example of a more general, and very important, phenomenon. As we shall see later in a number of different examples, the effect of a small force on a rotating system is often to make the axis of rotation precess—that is, revolve about some fixed direction. It will therefore be useful to consider here an alternative treatment of the problem which is of more general applicability.

If the magnetic field is weak, then to a first approximation the particle must move in an inverse square law orbit whose characteristics change slowly with time. Since the magnetic force is perpendicular to the velocity, it does no work. Hence the energy is a constant, and therefore so is the semi-major axis of the orbit. However, the angular momentum **J** does change with time according to the equation

$$\begin{aligned}\frac{d\mathbf{J}}{dt} &= \mathbf{r}\wedge\mathbf{F} = \frac{q}{c}\mathbf{r}\wedge(\mathbf{v}\wedge\mathbf{B})\\ &= \frac{q}{c}[(\mathbf{r}\cdot\mathbf{B})\mathbf{v}-(\mathbf{r}\cdot\mathbf{v})\mathbf{B}]. \qquad (5.26)\end{aligned}$$

To be specific, let us suppose that the magnetic field is in the z direction, $\mathbf{B} = B\mathbf{k}$, and that the particle is moving in a circular orbit of radius r in a plane inclined to the xy-plane at an angle α. (See Fig. 5.10.) It will be convenient to introduce three perpendicular unit vectors, $\mathbf{n}$ normal to the plane of the orbit, $\mathbf{a}$ in the direction of $\mathbf{k}\wedge\mathbf{n}$, and $\mathbf{b} = \mathbf{n}\wedge\mathbf{a}$. Then the angular momentum is

$$\mathbf{J} = m\mathbf{r}\wedge\mathbf{v} = mrv\mathbf{n}. \qquad (5.27)$$

If we specify the position of the particle in its orbit by the angle ψ between $\mathbf{a}$ and $\mathbf{r}$, then in terms of the axes $\mathbf{a}$, $\mathbf{b}$, $\mathbf{n}$, the components of $\mathbf{r}$, $\mathbf{v}$ and $\mathbf{B}$ are

$$\begin{aligned}\mathbf{r} &= (r\cos\psi,\ r\sin\psi,\ 0),\\ \mathbf{v} &= (-v\sin\psi,\ v\cos\psi,\ 0),\\ \mathbf{B} &= (0,\ B\sin\alpha,\ B\cos\alpha).\end{aligned}$$

Now in a circular orbit $\mathbf{r} \cdot \mathbf{v}$ is always 0. Hence the right side of (5.26) is

$$\frac{q}{c}(Br \sin \alpha \sin \psi)\mathbf{v} = \frac{qBrv}{c} \sin \alpha(-\sin^2 \psi, \sin \psi \cos \psi, 0). \tag{5.28}$$

Since the magnetic field is weak, the change in $\mathbf{J}$ in a single orbit will be small. Thus the oscillatory term in (5.28) is unimportant: it leads only to a very small periodic oscillation in $\mathbf{J}$. The important term is the one which leads to a *secular* variation of $\mathbf{J}$ (a steady change in one direction). It follows that we can replace (5.28) by

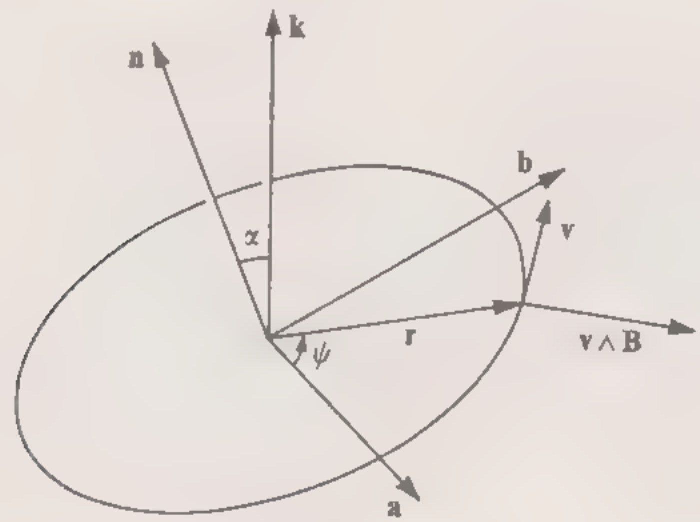


Fig. 5.10

its average value over a complete period; that is we replace $\sin^2\psi = \frac{1}{2}(1-\cos 2\psi)$ by $\frac{1}{2}$, and $\sin\psi \cos\psi$ by 0. This yields the equation

$$\frac{d\mathbf{J}}{dt} = \langle \mathbf{r} \wedge \mathbf{F} \rangle_{av} = -\frac{qBrv}{2c} \sin \alpha\, \mathbf{a}. \tag{5.29}$$

Thus the vector $\mathbf{J}$ will move in the direction of $-\mathbf{a}$. Since $d\mathbf{J}/dt$ is perpendicular to $\mathbf{J}$, the magnitude of $\mathbf{J}$ remains unchanged, while its direction precesses around the direction of $\mathbf{B}$. To find the rate of precession, we may rewrite the right side of (5.29) in terms of $\mathbf{J}$. From (5.27), we have

$$\mathbf{k} \wedge \mathbf{J} = mrv\mathbf{k} \wedge \mathbf{n} = mrv \sin \alpha\, \mathbf{a}.$$

Hence (5.29) may be written

$$\frac{d\mathbf{J}}{dt} = -\frac{qB}{2mc} \mathbf{k} \wedge \mathbf{J}. \tag{5.30}$$

This is the equation of a vector **J** rotating with angular velocity

$$\omega = -\frac{q\mathbf{B}}{2mc}.$$

Since the direction of **J** is that of the normal to the orbital plane, this agrees with our previous conclusion that the orbital plane precesses around the direction of the magnetic field, with a precessional angular velocity equal to the angular Larmor frequency.

Comparison with Current Loop. It is instructive to contrast this behaviour of a charged particle with the superficially similar case of a current loop. A circular current loop in the same position as the particle orbit would experience a similar force, producing a moment about the $-\mathbf{a}$-axis. The effect of the force, however, would be quite different. Since the loop itself (that is, its *mass*) is not rotating, the effect would be to rotate the loop about this axis, so that its normal tends to become aligned with the direction of **B**. (This is the familiar fact that a current loop behaves like a magnetic dipole. See Appendix B.)

To understand why a particle moving in an orbit behaves so differently, it may be helpful to consider, in place of a continuously acting force, a small impulsive blow delivered once every orbit, say at the farthest point above the xy-plane. The effect of such a blow is to give the particle a small component of velocity perpendicular to its original orbit. The next orbit will therefore be in a plane slightly tilted with respect to the original one, but reaching just as far from the xy-plane. The net effect is to make the orbit swing around the z-axis. The essential point here is that it is the velocity of the particle, rather than its position, which is changed instantaneously. Thus the effect of a small blow is not seen in a shift of the orbit at the point of the blow, but rather at a point 90° later.

We shall see in Chapter 10 that a rotating rigid body displays a very similar type of behaviour. If one applies a small force downwards on the rim of a rapidly rotating wheel, it is not that point which moves down, but a point 90° later. (This is very easy to verify with a freely spinning bicycle wheel.)

5.7 Summary

In problems involving a rotating body—particularly the earth—it is often convenient to use a rotating frame of reference. The equations of motion in such a frame contain additional terms representing apparent forces which arise because the frame is non-inertial. These are the centrifugal force, directed outwards from

the axis of rotation, and the velocity-dependent Coriolis force, $-2m\boldsymbol{\omega}\wedge\dot{\mathbf{r}}$. Because the magnetic force on a charged particle tends to produce rotation about the direction of the magnetic field, rotating frames are also useful in many problems involving a magnetic field.

PROBLEMS

1 A locomotive is travelling due north in colatitude θ along a straight level track with velocity v. Show that the ratio of the forces on the two rails is approximately

$$1+\frac{4\omega vh}{ga}\cos\theta,$$

where h is the height of the centre of gravity above the rails, and $2a$ is the width between the rails. Calculate this ratio for a velocity of 150 km/hr in latitude 45°N, assuming that $h = 2a$. Which rail experiences the larger force?

2 The wind velocity in colatitude θ is v. By considering the forces on a small volume of air, show that the pressure gradient required to balance the horizontal component of the Coriolis force, and thus maintain a constant wind direction, is

$$\frac{\mathrm{d}p}{\mathrm{d}x} = 2\rho\omega v\cos\theta,$$

where ρ is the density of the air. Evaluate this gradient in mbar/km for a wind velocity of 50 km/hr in latitude 30°N. (1 mbar $= 10^3$ dyne cm^{-2}; density of atmosphere $= 0{\cdot}0013$ g cm^{-3}.)

3 A beam of particles of charge q and velocity v is emitted from a point source, roughly parallel to a magnetic field $\mathbf{B}$, but with a small angular dispersion. Show that the effect of the field is to focus the beam to a point at a distance $z = 2\pi mcv/qB$ from the source. Calculate the focal distance for electrons of kinetic energy 8×10^{-10} erg in a magnetic field of 1 kG.

4 The angular velocity of the electron in the lowest Bohr orbit of the hydrogen atom is approximately 4×10^{16} s^{-1}. What is the largest magnetic field which may be regarded as small in this case, in the sense of §5.5? Determine the Larmor frequency in a field of 20 kG.

5 The orbit of an electron (charge $-e$) around a nucleus (charge Ze) is a circle of radius a in a plane perpendicular to a uniform magnetic field $\mathbf{B}$. By writing the equation of motion in a frame rotating with the electron, show that the angular velocity ω is given precisely by one of the roots of the equation

$$m\omega^2 - \frac{eB\omega}{c} - \frac{Ze^2}{a^3} = 0.$$

Verify that for small values of B, this agrees with §5.5. Evaluate the two roots if $B = 10^9$ G ($a = 5{\cdot}3\times10^{-9}$ cm).

6 A projectile is launched due north from a point in colatitude θ at an angle $\frac{1}{4}\pi$ to the horizontal, and aimed at a target whose distance is y (small

compared to R). Show that if no allowance is made for the effects of the Coriolis force, the projectile will miss its target by a distance

$$x = \omega\left(\frac{2y^3}{g}\right)^{1/2}(\cos\theta - \tfrac{1}{3}\sin\theta).$$

Evaluate this distance if $\theta = 45°$ and $y = 40$ km. Why is it that the deviation is to the east near the north pole, but to the west both at the equator and near the south pole? (Neglect atmospheric resistance.)

7 Solve the problem of a particle falling from height h above the equator by using an inertial frame, and verify that the answer agrees with that found using a rotating frame.

8 Solve the equation of motion of a particle falling freely from height h to second order in ω, and show that there is a deviation to the south (in the northern hemisphere) as well as that to the east. Calculate both components of the deviation for $h = 400$ m and latitude 50°N. How would an inertial observer interpret the southerly deviation?

9 Find the equation of motion for a particle in a frame rotating with *variable* angular velocity $\boldsymbol{\omega}$, and show that there is another apparent force of the form $-m\dot{\boldsymbol{\omega}} \wedge \mathbf{r}$. Discuss the physical origin of this force.

Chapter 6 Potential Theory

In this chapter, we shall discuss the problem of determining the gravitational or electrostatic force acting on a particle from a knowledge of the positions of other masses or charges. We return here to the convention whereby $\dot{\mathbf{r}} = d\mathbf{r}/dt$. We shall not use rotating frames again until Chapter 10.

6.1 Gravitational and Electrostatic Potentials

The gravitational potential energy of a particle of mass m moving in the field of a fixed mass m' at $\mathbf{r}'$ is $-Gmm'/|\mathbf{r}-\mathbf{r}'|$. If we have several masses m_j located at the points $\mathbf{r}_j$, then the potential energy is the sum

$$V(\mathbf{r}) = -\sum_j \frac{Gmm_j}{|\mathbf{r}-\mathbf{r}_j|}.$$

(The fact that the potential energies add follows from the additive property of forces.)

Since the mass m appears only as an overall factor, we may define the *gravitational potential* $\Phi(\mathbf{r})$ to be the potential energy per unit mass,

$$V(\mathbf{r}) = m\Phi(\mathbf{r}), \tag{6.1}$$

so that

$$\Phi(\mathbf{r}) = -\sum_j \frac{Gm_j}{|\mathbf{r}-\mathbf{r}_j|}. \tag{6.2}$$

Note that with this definition Φ is always negative.*

The acceleration of the particle is given by

$$m\ddot{\mathbf{r}} = \mathbf{F} = -\nabla V(\mathbf{r}) = -m\nabla\Phi(\mathbf{r}).$$

Since this acceleration is independent of the mass m, we may define the *gravitational acceleration* or *gravitational field* $\mathbf{g}(\mathbf{r})$ by

$$\mathbf{g}(\mathbf{r}) = -\nabla\Phi(\mathbf{r}). \tag{6.3}$$

Thus, equations (6.2) and (6.3) are all that is needed to calculate the acceleration induced in a particle by a given distribution of masses.

* The potential is sometimes defined to be *minus* the potential energy per unit mass, and therefore positive. We prefer, however, to retain the direct correspondence between potential and potential energy, so that a particle tends to move towards regions of lower potential.

The electrostatic case is very similar. We define the *electrostatic potential* $\phi(\mathbf{r})$ to be the electrostatic potential energy per unit charge,

$$V(\mathbf{r}) = q\phi(\mathbf{r}). \tag{6.4}$$

The potential due to charges q_j at $\mathbf{r}_j$ is*

$$\phi(\mathbf{r}) = \sum_j \frac{q_j}{|\mathbf{r}-\mathbf{r}_j|}. \tag{6.5}$$

The acceleration of a particle is given by

$$m\ddot{\mathbf{r}} = q\mathbf{E},$$

where the electric field $\mathbf{E}(\mathbf{r})$ is defined by

$$\mathbf{E} = -\nabla\phi(\mathbf{r}). \tag{6.6}$$

Note that the acceleration depends only on the charge-to-mass ratio q/m.†

Unlike the gravitational potential, the electrostatic potential may have either sign, because both positive and negative charges exist.

We can now forget about the particle at $\mathbf{r}$, and calculate the potential from given information about the positions of the masses or charges.

6.2 The Dipole and Quadrupole

The *electric dipole* consists of two equal and opposite charges, q and $-q$, placed close together, say at $\mathbf{a}$ and at the origin respectively. (See Fig. 6.1.) The potential is

$$\phi(\mathbf{r}) = \frac{q}{|\mathbf{r}-\mathbf{a}|} - \frac{q}{r}. \tag{6.7}$$

We shall assume that $a \ll r$, so that we may expand by the binomial theorem. Since we shall need the result later, we evaluate the terms up to order a^2, but neglect a^3. If θ is the angle between $\mathbf{a}$ and $\mathbf{r}$, then

$$|\mathbf{r}-\mathbf{a}|^2 = r^2 - 2ra\cos\theta + a^2.$$

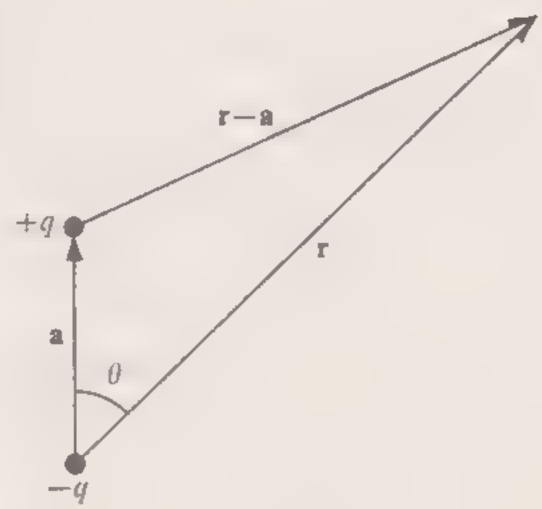


Fig. 6.1

* In SI units, this equation (and all subsequent equations relating the electrostatic potential or field to the charge, dipole moment, etc.) requires an extra factor $1/4\pi\varepsilon_0$ on the right-hand side.

† The corresponding factor in the gravitational case is the ratio of gravitational to inertial mass, which is of course a constant. We could make the correspondence even closer by defining the mass m_g in gravitational units to be $G^{1/2}m$, so that the potential energy has the form $-m_g m'_g/|\mathbf{r}-\mathbf{r}'|$.

Hence

$$\frac{1}{|\mathbf{r}-\mathbf{a}|} = \frac{1}{r}\left(1-2\frac{a}{r}\cos\theta+\frac{a^2}{r^2}\right)^{-1/2}$$

$$= \frac{1}{r}\left[1-\frac{1}{2}\left(-2\frac{a}{r}\cos\theta+\frac{a^2}{r^2}\right) + \frac{3}{8}\left(-2\frac{a}{r}\cos\theta+\frac{a^2}{r^2}\right)^2 - \ldots\right]$$

$$= \frac{1}{r}+\frac{a}{r^2}\cos\theta+\frac{a^2}{r^3}\left(\tfrac{3}{2}\cos^2\theta-\tfrac{1}{2}\right)+\ldots.^* \qquad (6.8)$$

In vector notation, we can write (6.8) as

$$\frac{1}{|\mathbf{r}-\mathbf{a}|} = \frac{1}{r}+\frac{\mathbf{a}\cdot\mathbf{r}}{r^3}+\frac{3(\mathbf{a}\cdot\mathbf{r})^2-a^2r^2}{2r^5}+\ldots. \qquad (6.9)$$

We now return to (6.7). Keeping only the linear term in a, and neglecting a^2, we can write it as

$$\phi(\mathbf{r}) = \frac{\mathbf{d}\cdot\mathbf{r}}{r^3} = \frac{d}{r^2}\cos\theta, \qquad (6.10)$$

where $\mathbf{d}$ is the *electric dipole moment*, $\mathbf{d} = q\mathbf{a}$. The corresponding electric field, given by (6.6), has the spherical polar components (see Appendix A (A.42))†

$$E_r = -\frac{\partial\phi}{\partial r} = \frac{2d\cos\theta}{r^3},$$

$$E_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{d\sin\theta}{r^3}, \qquad (6.11)$$

$$E_\varphi = -\frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi} = 0.$$

This field is illustrated in Fig. 6.2, in which the solid lines are the field lines, drawn in the direction of $\mathbf{E}$, and the dashed lines are the equipotential surfaces ϕ = constant. Note that the two always intersect at right angles.

* The general term in this series is of the form $(a^l/r^{l+1})P_l(\cos\theta)$, where P_l is a polynomial known as the *Legendre polynomial* of degree l. However, we shall only need the terms up to $l = 2$.

†To avoid any possible risk of confusion, we denote the potential and the polar angle by the distinct symbols ϕ and φ.

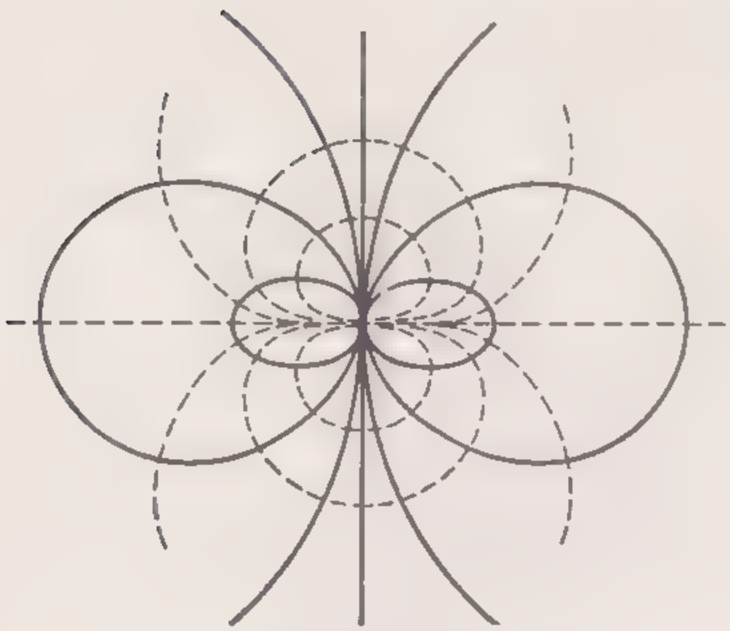


Fig. 6.2

We can repeat the process of putting two charges (or *monopoles*) together to form a dipole, by putting two dipoles together to form a *quadrupole*. If we place oppositely oriented dipoles, of dipole moments **d** and $-$**d**, at **a** and at the origin, then the potential is

$$\phi(\mathbf{r}) = \frac{\mathbf{d}\cdot(\mathbf{r}-\mathbf{a})}{|\mathbf{r}-\mathbf{a}|^3} - \frac{\mathbf{d}\cdot\mathbf{r}}{r^3}.$$

Now, neglecting terms of order a^2, we have

$$\frac{1}{|\mathbf{r}-\mathbf{a}|^3} \approx \frac{1}{r^3}\left(1-2\frac{\mathbf{a}\cdot\mathbf{r}}{r^2}\right)^{-3/2} \approx \frac{1}{r^3}+3\frac{\mathbf{a}\cdot\mathbf{r}}{r^5}.$$

Hence, to this approximation,

$$\phi(\mathbf{r}) = \frac{3(\mathbf{d}\cdot\mathbf{r})(\mathbf{a}\cdot\mathbf{r})-(\mathbf{d}\cdot\mathbf{a})r^2}{r^5}. \tag{6.12}$$

In the special case where the dipoles are placed end-on, so that **d** is parallel to **a**, we may take this common direction to be the z-axis, and obtain

$$\phi(\mathbf{r}) = \frac{Q}{4r^3}(3\cos^2\theta - 1), \tag{6.13}$$

where Q is the *electric quadrupole moment*, $Q = 4da$. (The factor of 4 is purely conventional, and serves to simplify some of the later

formulae.) The corresponding electric field is

$$E_r = \frac{3Q}{4r^4}(3\cos^2\theta - 1),$$
$$E_\theta = \frac{3Q}{2r^4}\cos\theta\sin\theta, \qquad (6.14)$$
$$E_\varphi = 0.$$

It is illustrated in Fig. 6.3.

6.3 Spherical Charge Distributions

When we have a continuous distribution of charge (or mass), we must replace the sum in (6.5) by an integral,

$$\phi(\mathbf{r}) = \iiint \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}^3\mathbf{r}', \qquad (6.15)$$

where $\rho(\mathbf{r}')$ is the *charge density*, and $\mathrm{d}^3\mathbf{r}'$ denotes the element of volume, $\mathrm{d}^3\mathbf{r}' = \mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z'$. In spherical polars, the volume element is the product of the elements of length in the three co-ordinate directions, namely (see §3.4)

$$\mathrm{d}^3\mathbf{r}' = r'^2\,\mathrm{d}r'\sin\theta'\,\mathrm{d}\theta'\,\mathrm{d}\varphi'.$$

Note that θ' ranges from 0 to π, and φ' from 0 to 2π.

We shall consider in this section spherically symmetric distributions of charge, for which ρ is a function only of the radial coordinate r'. We consider first a uniform spherical shell of charge density ρ, radius a and thickness $\mathrm{d}a$. Choosing the direction of $\mathbf{r}$ to be the z-axis, we can write (6.15) as

$$\phi(\mathbf{r}) = \rho a^2\,\mathrm{d}a \iint \frac{\sin\theta'\,\mathrm{d}\theta'\,\mathrm{d}\varphi'}{(r^2 - 2ar\cos\theta' + a^2)^{1/2}}.$$

The φ' integration gives a factor of 2π. The θ' integration can easily be performed by the substitution $w = \cos\theta'$, and yields

$$\phi(\mathbf{r}) = 2\pi\rho a^2\,\mathrm{d}a\,\frac{(r+a) - |r-a|}{ar}.$$

Thus, in terms of the total charge $\mathrm{d}q = 4\pi\rho a^2\,\mathrm{d}a$ of the spherical shell, we obtain

$$\phi(\mathbf{r}) = \frac{\mathrm{d}q}{r}, \qquad r > a,$$
$$\phi(\mathbf{r}) = \frac{\mathrm{d}q}{a}, \qquad r < a. \qquad (6.16)$$

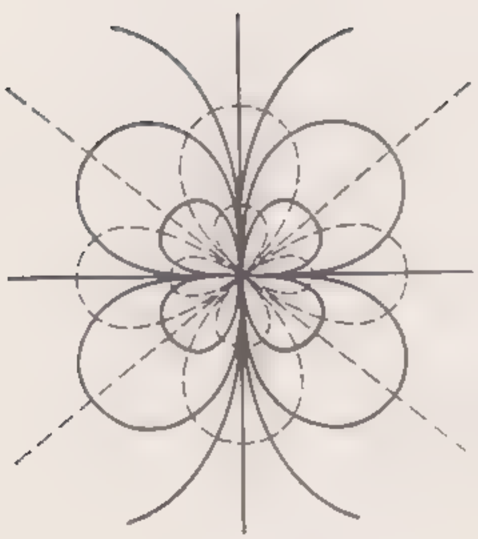


Fig. 6.3

Outside the shell, the potential is the same as that of a charge dq concentrated at the origin, and the electric field is $\mathbf{E} = \hat{\mathbf{r}}dq/r^2$. Inside, the potential is constant, and the electric field vanishes.

It is now easy to find the potential of any spherically symmetric distribution, by summing over all the spherical shells. It is clear that the electric field at a distance r from the centre is equal to that of a point charge located at the origin, whose magnitude is the total charge enclosed within a sphere of radius r. (A similar result holds in the gravitational case.)

In particular, for a uniformly charged sphere of radius a, the potential outside the sphere is just q r, where q is the total charge.

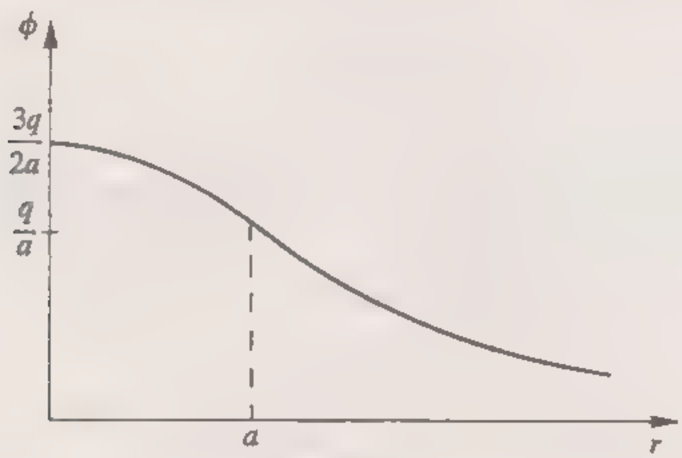


Fig. 6.4

Inside, we must separate the contributions from the regions $r'<r$ and $r'>r$, and obtain

$$\phi(\mathbf{r}) = \int_0^r \frac{4\pi\rho r'^2\, dr'}{r} + \int_r^a 4\pi\rho r'\, dr'.$$

Performing the integrations, we find

$$\begin{aligned} \phi(\mathbf{r}) &= \frac{q}{r}, & r>a, \\ \phi(\mathbf{r}) &= q\left(\frac{3}{2a} - \frac{r^2}{2a^3}\right), & r<a. \end{aligned} \tag{6.17}$$

This potential is illustrated in Fig. 6.4. The corresponding electric field is

$$\begin{aligned} \mathbf{E} &= \hat{\mathbf{r}}\frac{q}{r^2}, & r>a, \\ \mathbf{E} &= \hat{\mathbf{r}}\frac{qr}{a^3}, & r<a. \end{aligned} \tag{6.18}$$

Thus, outside the sphere, the field obeys the inverse square law. Inside, it increases linearly from zero at the centre to the surface value q/a^2.

As far as its effect on outside bodies is concerned, any spherically symmetric charge or mass distribution may be replaced by a point charge or mass located at the centre. This is the justification for treating the sun, for example, as a point mass in discussing the motions of the planets. Of course, any deviation from spherical symmetry will lead to a modification in the inverse square law. We shall discuss the nature of this correction later.

6.4 Expansion of Potential at Large Distances

It is only for a few simple cases that we can calculate the potential exactly. In general, we must use approximation methods. In particular, we are often interested in the form of the potential at distances from the charge distribution which are large compared to its dimensions. Then, in (6.5) or (6.15), r is much larger than r_j or r', and we may use the series expansion (6.9) to obtain an expansion of the potential in powers of r'/r,

$$\phi(\mathbf{r}) = \phi_0(\mathbf{r})+\phi_1(\mathbf{r})+\phi_2(\mathbf{r})+\dots . \tag{6.19}$$

The leading term is

$$\phi_0(\mathbf{r}) = \frac{q}{r}, \tag{6.20}$$

where q is the total charge

$$q = \sum_j q_j \quad \text{or} \quad q = \iiint \rho(\mathbf{r}')\, d^3\mathbf{r}'. \tag{6.21}$$

Thus at very great distances only the total charge is important, not the shape of the distribution. This is the potential we should get for a spherically symmetric distribution. Thus the further terms in the series (6.19) may be regarded as measuring the deviations from spherical symmetry.

The next term in (6.19) is

$$\phi_1(\mathbf{r}) = \frac{\mathbf{d}\cdot\mathbf{r}}{r^3}, \tag{6.22}$$

where $\mathbf{d}$ is the *dipole moment*, defined by

$$\mathbf{d} = \sum_j q_j\mathbf{r}_j \quad \text{or} \quad \mathbf{d} = \iiint \rho(\mathbf{r}')\mathbf{r}'\, d^3\mathbf{r}'. \tag{6.23}$$

If we shift the origin through a distance **R**, the total charge is clearly unaffected, but the dipole moment changes to

$$\mathbf{d}^* = \sum_j q_j(\mathbf{r}_j - \mathbf{R}) = \mathbf{d} - q\mathbf{R}. \tag{6.24}$$

Thus if the total charge is non-zero, we can always make the dipole moment vanish by choosing the origin at the *centre of charge*

$$\mathbf{R} = \frac{\sum q_j \mathbf{r}_j}{\sum q_j} = \frac{\mathbf{d}}{q}.$$

In this case, the dipole moment gives us information about the position of the centre of charge. (In the gravitational case, the total mass can never vanish, and we can always arrange that the gravitational 'dipole moment' be zero by choosing our origin at the centre of mass.) If the total charge q is zero, then by (6.24) the dipole moment is independent of the choice of origin—the electric dipole is an example of this case. For, if we shift the origin to **R**, the dipole moment $q\mathbf{a}$ changes to $q(\mathbf{a}-\mathbf{R})-q(-\mathbf{R}) = q\mathbf{a}$.

The 'quadrupole' term in the potential (6.19), obtained by substituting the third term of (6.9) into (6.15), is

$$\phi_2(\mathbf{r}) = \iiint \rho(\mathbf{r}') \frac{3(\mathbf{r}\cdot\mathbf{r}')^2 - r^2 r'^2}{2r^5} \, \mathrm{d}^3\mathbf{r}'. \tag{6.25}$$

One can write this expression in a form similar to (6.12). However, we shall not consider the general case, but restrict the discussion to the case of axial symmetry, where $\rho(\mathbf{r}')$ depends on r' and θ' (or z' and ρ'), but not on φ'. This case is of particular interest in connection with the gravitational potential of the earth, which is to a good approximation axially symmetric, though flattened at the poles. We shall show that in the case of axial symmetry, (6.25) can be written in the form (6.13) with a suitably defined quadrupole moment Q.

To do this, we examine the numerator of the integrand in (6.25). Written out in terms of components, it reads

$$\begin{aligned} &x^2(2x'^2 - y'^2 - z'^2) + y^2(2y'^2 - x'^2 - z'^2) \\ &\quad + z^2(2z'^2 - x'^2 - y'^2) + 6xyx'y' + 6xzx'z' + 6yzy'z'. \end{aligned} \tag{6.26}$$

Now, because of the axial symmetry, any integral involving an odd power of x' or y' will vanish, for example

$$\iiint \rho(\mathbf{r}') x'z' \, \mathrm{d}^3\mathbf{r}' = 0,$$

because the contributions from $x'>0$ and $x'<0$ exactly cancel. Moreover, it also follows that $\iiint \rho(\mathbf{r}')x'^2\,\mathrm{d}^3\mathbf{r}' = \iiint \rho(\mathbf{r}')y'^2\,\mathrm{d}^3\mathbf{r}'$, since the x- and y-axes are in no way distinguished from one another. Hence we can replace $2x'^2-y'^2-z'^2$ by $\frac{1}{2}(x'^2+y'^2)-z'^2$ without affecting the value of the integral. When we do this, we see that the three remaining terms of (6.26) all involve the same integral, namely

$$Q = \iiint \rho(\mathbf{r}')(2z'^2-x'^2-y'^2)\,\mathrm{d}^3\mathbf{r}', \tag{6.27}$$

and therefore that (6.25) can be written

$$\phi_2(\mathbf{r}) = \frac{2z^2-x^2-y^2}{4r^5}\,Q = \frac{Q}{4r^3}\,(3\cos^2\theta-1). \tag{6.28}$$

This is identical with the potential (6.13) of an axially symmetric quadrupole. The quantity Q defined by (6.27) is the *quadrupole moment* of the distribution.

Thus we see that a better approximation than treating a charge distribution as a point charge is to treat it as a point charge plus an electric quadrupole of moment given by (6.27). For a spherically symmetric charge distribution, $Q = 0$, because the integrals over x'^2, y'^2 and z'^2 all yield the same value. The value of Q may be regarded as a measure of the flattening of the distribution. For a uniform distribution of positive charge, Q is positive if the shape is *prolate* (egg-shaped, with the z-axis longer than the others), and negative if it is *oblate*. In particular, for a uniformly charged *spheroid* (ellipsoid of revolution), with semi-axes a, a, c, we can evaluate the integral (6.27) explicitly. We shall have occasion to evaluate a very similar integral in connection with moments of inertia in Chapter 9, and therefore omit the details here. The result is

$$Q = \tfrac{2}{5}q(c^2-a^2). \tag{6.29}$$

6.5 The Shape of The Earth

The earth is approximately an oblate spheroid whose equatorial radius a exceeds its polar radius c by about 21·5 km. The *oblateness*

$$\varepsilon = \frac{a-c}{a} \tag{6.30}$$

is just over 1/300. Its gravitational potential is therefore not precisely the inverse square law potential, $-GM/r$. The most important

correction is the 'quadrupole' term

$$\Phi_2(\mathbf{r}) = -\frac{GQ}{4r^3}(3\cos^2\theta - 1).$$

Since the earth is oblate, Q is actually negative.

If the density of the earth were uniform, the quadrupole moment would be given simply by (6.29),

$$Q = -\tfrac{2}{5}M(a^2 - c^2) \approx -\tfrac{4}{5}Ma^2\varepsilon.$$

In fact, the density is considerably greater near the centre than at the surface, so that large values of r' contribute proportionately less to the integral in (6.27). Thus we must expect Q to be rather smaller than this value. It will be convenient to write

$$Q = -\tfrac{4}{5}Ma^2\lambda,$$

where λ is a dimensionless parameter, somewhat less than ε. Then the gravitational potential becomes

$$\Phi(\mathbf{r}) = -\frac{GM}{r} + \frac{GMa^2\lambda}{5r^3}(3\cos^2\theta - 1). \tag{6.31}$$

We shall see that both ε and λ can be determined from observation.

The flattening of the earth is in fact a consequence of its rotation. Over long periods of time, the earth is capable of plastic deformation, and behaves more nearly like a viscous liquid than a solid, although of course its crust does have some rigidity. It could not preserve its shape for long unless it were at least approximately in equilibrium under the combined gravitational and centrifugal forces.

Now, as in the problem discussed at the end of §5.3, the surface of a liquid in equilibrium under conservative forces must be an equipotential surface, for otherwise there would be a tendency for the liquid to flow towards regions of lower potential. We can include the centrifugal force by adding to the potential a term

$$\Phi_{\text{cent.}}(\mathbf{r}) = -\tfrac{1}{2}\omega^2(x^2 + y^2) = -\tfrac{1}{2}\omega^2 r^2 \sin^2\theta.$$

The equipotential equation is

$$\Phi(\mathbf{r}) + \Phi_{\text{cent.}}(\mathbf{r}) = \text{constant}, \tag{6.32}$$

with Φ given by (6.31). Thus, equating the values at the pole and on the equator, we obtain the equation

$$-\frac{GM}{c} + \frac{2GMa^2\lambda}{5c^3} = -\frac{GM}{a} - \frac{GM\lambda}{5a} - \tfrac{1}{2}\omega^2 a^2.$$

Since ε and λ are both small quantities, we may neglect ε^2 and $\varepsilon\lambda$. Thus, in the quadrupole term, we may neglect the difference between a and c. In the inverse square law term, we can write $1/c \approx (1/a)(1+\varepsilon)$. Hence to first order we find

$$-\frac{GM\varepsilon}{a}+\frac{3GM\lambda}{5a} = -\tfrac{1}{2}\omega^2 a^2,$$

or, writing $g_0 = GM/a^2$,

$$\frac{\omega^2 a}{g_0} = 2\varepsilon - \tfrac{8}{5}\lambda. \tag{6.33}$$

If we made the approximation of treating the earth as of uniform density, and set $\lambda = \varepsilon$, then this equation would determine the oblateness in terms of the angular velocity. This yields $\varepsilon \approx 1/230$, which is appreciably larger than the observed value.

On the other hand, if we regard ε and λ as independent, then we need another relation to fix them both. Such a relation can be found from the measured values of g. (See §5.3.) Now, the gravitational field corresponding to the potential (6.31) is

$$\begin{aligned} g_r &= -\frac{GM}{r^2}+\frac{3GMa^2\lambda}{5r^4}(3\cos^2\theta-1), \\ g_\theta &= \frac{6GMa^2\lambda}{5r^4}\cos\theta\sin\theta. \end{aligned} \tag{6.34}$$

Note that g_θ is directed away from the poles, and towards the equator, as one might expect from thinking of it as due to the attraction of the equatorial bulge. The inward radial acceleration $-g_r$ is decreased at the poles, and increased at the equator, so that the difference between the two values is not as large as the inverse square law alone would predict.

At the poles, and on the equator, $g_\theta = 0$, and the magnitude of $\mathbf{g}$ is equal to $-g_r$. Thus we find

$$g_{\text{pole}} = \frac{GM}{c^2}-\frac{6GMa^2\lambda}{5c^4} \approx g_0(1+2\varepsilon-\tfrac{6}{5}\lambda),$$

$$g_{\text{eq.}} = \frac{GM}{a^2}+\frac{3GM\lambda}{5a^2} \approx g_0(1+\tfrac{3}{5}\lambda),$$

making the same approximations as before. Thus, for the difference $\Delta g = g_{\text{pole}} - g_{\text{eq.}}$, we find the expression

$$\frac{\Delta g}{g_0} = 2\varepsilon - \tfrac{9}{5}\lambda. \tag{6.35}$$

For the value including the centrifugal term, $\Delta g^* = \Delta g + \omega^2 a$, we have to add (6.33) to (6.35). This yields

$$\frac{\Delta g^*}{g_0} = 4\varepsilon - 3\lambda. \tag{6.36}$$

Using the known values quoted in (5.17) and (5.18), we may solve the equations (6.33) and (6.36) for ε and λ. This yields $\varepsilon = 0{\cdot}0034$ and $\lambda = 0{\cdot}0028$. As expected, we find that λ is somewhat less than ε. The relation between the two is determined by the density distribution within the earth. If we knew this distribution with sufficient accuracy, we could in fact calculate λ from ε.

One consequence of the fact that the earth's surface is approximately an equipotential surface should be noted. Because of the general property that the field is always perpendicular to the equipotential surfaces, it means that the observed $\mathbf{g}^*$, which is the field corresponding to $\Phi + \Phi_{\text{cent.}}$, is always perpendicular to the earth's surface. Thus, although a plumb line does not point towards the earth's centre, it is perpendicular to the surface at that point.

Satellite Orbits. The 'quadrupole' term in the earth's gravitational field has two important effects on the orbit of a close artificial satellite. Indeed, observations of such orbits provide the most reliable means of measuring its magnitude (and those of the even smaller higher order correction terms). They have revealed quite substantial deviations both from spheroidal shape and from hydrostatic equilibrium: the earth's surface is only rather approximately an equipotential.

The first effect of the quadrupole term, arising mainly from the deviation of the radial component of $\mathbf{g}$ from the inverse square law, is a precession of the major axis of the orbit within the orbital plane. (Compare Chapter 4, Problem 13.) The major axis precesses in the forward direction for orbits of small inclination to the equator, and in the retrograde direction for orbits with inclination greater than $\sin^{-1}(4/5)^{1/2} = 63{\cdot}4°$. (This difference is a reflection of the differing sign of the quadrupole field on the equator and near the poles.)

The second effect occurs because the force is no longer precisely central, so that the angular momentum $\mathbf{J}$ changes with time according to

$$\frac{d\mathbf{J}}{dt} = m\mathbf{r} \wedge \mathbf{g}. \tag{6.37}$$

Thus, as in the case of the Larmor effect discussed in §5.6, the orbital plane precesses around the direction of the earth's axis. The rate of precession may be calculated by a method very similar to the one used there. (See Problem 10.) The precession is in fact greatest for

orbits of small inclination, and is zero for an orbit passing over the poles. For a close satellite with small inclination, it can be nearly 10° per day. It is always in a retrograde sense (opposite to the direction of revolution of the satellite).

Both effects are strongly dependent on the radius of the satellite orbit. In fact, the rate of precession decreases like $r^{-7/2}$. At the radius of the moon's orbit, the precessional angular velocity is only a few seconds of arc per year. (The moon's orbit does precess, at about 19° per year, but this is a consequence of the non-uniformity of the sun's gravitational field, not of the shape of the earth.)

6.6 The Tides

Tidal forces arise because the gravitational attraction of the moon, and to a lesser extent that of the sun, is not uniform over the surface of the earth. The attraction is stronger than average on the side of the earth facing the moon, and weaker than average on the far side, so that there is a tendency for the earth to be elongated along the line of centres.

Let $\mathbf{a}$ be the position of the moon relative to the earth's centre, and consider a point $\mathbf{r}$ on the earth. The potential at this point is (see Fig. 6.5)

$$\Phi(\mathbf{r}) = -\frac{Gm}{|\mathbf{r}-\mathbf{a}|}, \tag{6.38}$$

where m is the mass of the moon. Now, since $r \ll a$, we may expand in powers of r/a. (This is the reverse of the situation encountered previously, where r was much larger than a.) Taking the direction of the moon to be the z direction, and using (6.8), we obtain

$$\Phi(\mathbf{r}) = -Gm\left[\frac{1}{a}+\frac{r}{a^2}\cos\theta+\frac{r^2}{a^3}\left(\tfrac{3}{2}\cos^2\theta-\tfrac{1}{2}\right)+\ldots\right].$$

The first term is a constant, and does not yield any force. The linear term may be written

$$\Phi_1(\mathbf{r}) = -\frac{Gm}{a^2}z.$$

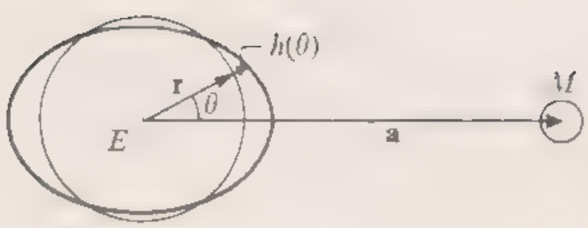


Fig. 6.5

It yields a uniform gravitational acceleration Gm/a^2 directed towards the moon. This term therefore describes the major effect of the moon's gravitational force, which is to accelerate the earth as a whole. It is irrelevant in discussing the phenomenon of the tides, since we are interested in motion relative to the centre of the earth.

The important term for our purposes is the quadratic term, which leads to a gravitational field

$$g_r = \frac{Gmr}{a^3}(3\cos^2\theta - 1),$$
$$g_\theta = -\frac{3Gmr}{a^3}\cos\theta\sin\theta. \tag{6.39}$$

This field is directed outwards along the z-axis, and inwards in the xy-plane, as was expected.

It should be noted that the field described by (6.39) is much weaker even than the corrections to the earth's field discussed in the preceding section. (If it were not, it would show up very obviously in measurements of g.) It is easy to compute its magnitude. It is smaller than $g_0 = GM/r^2$ by the factor mr^3/Ma^3. For the moon, $m/M = 1/81$ and $r/a = 1/60$, so that this factor is

$$\frac{mr^3}{Ma^3} = 5{\cdot}7\times 10^{-8}. \tag{6.40}$$

If we denote the mass of the sun by m', and its mean distance by a', then the corresponding values are $m'/M = 3{\cdot}3\times 10^5$ and $r/a' = 4{\cdot}3\times 10^{-5}$. Thus the factor in that case is

$$\frac{m'r^3}{Ma'^3} = 2{\cdot}6\times 10^{-8}. \tag{6.41}$$

By a remarkable coincidence (unique in the solar system), these two fields are of roughly the same order of magnitude. The effect of the sun is rather less than half that of the moon.

The only reason why these very small fields can lead to significant effects is that they change with time. As the earth rotates, the angle θ at any point on its surface varies. Thus the field described by (6.39) oscillates periodically with time. Because of the symmetry of (6.39) in the central plane, the main term has an oscillation period of 12 hours in the case of the sun, and slightly more for the moon, because of its changing position. This explains the most noticeable feature of the tides—their twice-daily periodicity. Unless the moon is directly over the equator, however, a point on the earth which passes directly under it will not also pass directly opposite, so that the two daily tides may be of unequal height. In other words, there is an additional term in the field with a period of 24 rather than 12 hours.

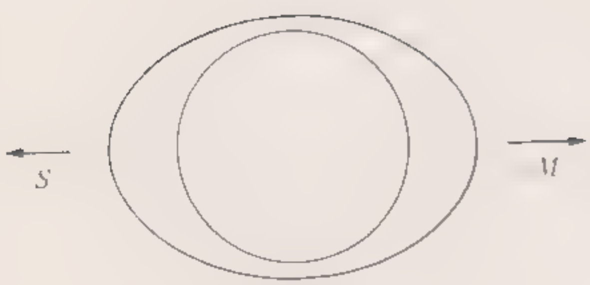
At new moon or full moon, the sun and moon are acting in the same direction, and the tides are unusually high; these are the *spring tides*. On the other hand, at the first and third quarters,

when the sun and moon are at right angles to each other, their effects partially cancel, and we have the low *neap tides*. (See Fig. 6.6.) Measurements of the relative heights of the tides at these times provided one of the earliest methods of estimating the mass of the moon. (A more accurate method will be discussed in the next chapter.)

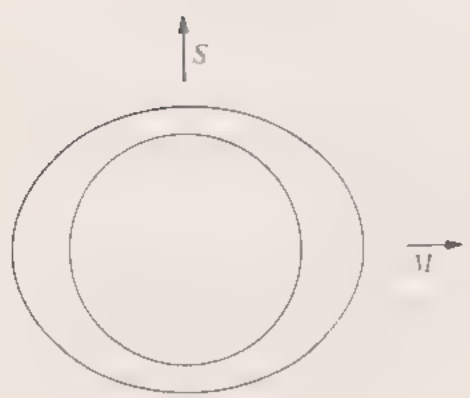
To calculate the height of the tides produced by the gravitational field (6.39), we have to solve the problem of a liquid moving under a periodic force. This is a complicated problem which involves a large number of factors, including for example the depth and shape of the sea bed. However, we can obtain some idea of the magnitude of the effect by assuming that the natural periods of oscillation are short compared to the rotation period of the earth. Under these circumstances, the rotation is so slow that the water on the earth's surface can reach equilibrium under the combined forces of earth and moon. The problem is then one in hydrostatics rather than hydrodynamics, and quite easy to solve.

In this equilibrium problem, the surface of the water must be an equipotential surface. We shall calculate the height $h(\theta)$ through which it is raised, as a function of the angle θ to the moon's direction. (See Fig. 6.5.) Since $h(\theta)$ is certainly small, the change in the earth's gravitational potential is approximately $g_0 h(\theta)$. This change must be balanced by the potential due to the moon, so that

$$g_0 h(\theta) = \frac{Gmr^2}{a^3}\left(\tfrac{3}{2}\cos^2\theta - \tfrac{1}{2}\right).$$



a Spring tides



b Neap tides

Fig. 6.6

Hence, using $g_0 = GM/r^2$, we find

$$h(\theta) = h_0\left(\tfrac{3}{2}\cos^2\theta - \tfrac{1}{2}\right), \tag{6.42}$$

where

$$h_0 = \frac{mr^3}{Ma^3}\,r. \tag{6.43}$$

The factor multiplying r is just the one we calculated earlier. Thus, using $r = 6370$ km, we find for the moon $h_0 = 36$ cm, and for the sun $h_0' = 16$ cm.

These values may seem rather small, but the drastic nature of our approximation must be remembered. The moon's attraction at any point on the earth is a periodic force, and its effect may be greatly enhanced by the phenomenon of resonance, discussed in §2.6. The natural periods of oscillation of a body of water depend on a variety of factors, including its size, shape and depth, and there are critical values for which one of the periods is close to 12 hours. Then,

because the damping produced by tidal friction is normally quite small except in very shallow water, very large tides can be set up.

Another effect of resonance is to delay the oscillations, so that they are not precisely in phase with the applied force. At resonance, the phase lag is a quarter period (about 3 hours in this case). Thus we should not normally expect the times of high tide to coincide with the times when the moon is overhead, or directly opposite. In practice, the times are determined in a very complicated way by the actual shapes of the oceans.

6.7 The Field Equations

It is often convenient to obtain the electric or gravitational potential by solving a differential equation rather than performing an integration. Although we shall not in fact need to use this technique in this book, we include a short discussion of it because of its very important role in more advanced treatments of mechanics and electromagnetic theory.

To find the relevant equations, we consider first a single charge q located at the origin. The electric field is then

$$\mathbf{E} = \frac{q}{r^2}\hat{\mathbf{r}}.$$

Now let us consider a closed surface S surrounding the charge, and evaluate the surface integral of the component of E normal to the surface,

$$\iint_S \mathbf{E}\cdot\mathbf{n}\,\mathrm{d}S = q\iint_S \frac{\hat{\mathbf{r}}\cdot\mathbf{n}}{r^2}\,\mathrm{d}S,$$

where $\mathrm{d}S$ is the element of surface area, and $\mathbf{n}$ is a unit vector normal to the surface, directed outwards. (See Fig. 6.7.) If α is the angle between $\mathbf{r}$ and $\mathbf{n}$, then $\hat{\mathbf{r}}\cdot\mathbf{n}\,\mathrm{d}S = \mathrm{d}S\cos\alpha$ is the projection of the area $\mathrm{d}S$ on a plane normal to the radius vector $\mathbf{r}$. Hence it follows from the definition of solid angle (see (4.44)) that $\hat{\mathbf{r}}\cdot\mathbf{n}\,\mathrm{d}S/r^2$ is equal to the solid angle $\mathrm{d}\Omega$ subtended at the origin by the element of area $\mathrm{d}S$. Hence

$$\iint_S \mathbf{E}\cdot\mathbf{n}\,\mathrm{d}S = q\iint \mathrm{d}\Omega = 4\pi q, \tag{6.44}$$

since the total solid angle subtended by a closed surface around the origin is 4π.

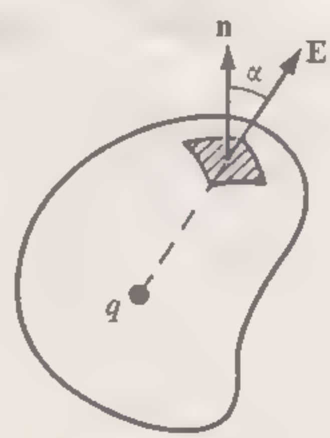


Fig. 6.7

We have implicitly assumed that each radial line cuts the surface only once. However, we can easily remove this restriction. Evidently, a radial line must always cut it in an odd number of points. (See Fig. 6.8.) Moreover, the contributions to the surface integral from points where it goes into, rather than out of, the surface are all of opposite sign (because $\hat{\mathbf{r}}\cdot\mathbf{n} = \cos\alpha$ is negative at these points). Thus all the terms but one will cancel, and we obtain the same answer (6.44).

Similarly, if we consider a surface S which does not enclose the origin, then we can see that the surface integral will vanish. For each radial line cuts S in an even number of points, half in each direction, so that all the contributions will cancel.

It is now clear how we may generalize this result to an arbitrary distribution of charges. Any charge q located anywhere within a closed surface S will contribute to the surface integral a term $4\pi q$, while charges outside contribute nothing. Therefore, the value of the integral is 4π times the total charge enclosed within the surface. If we have a continuous distribution of charge, with charge density $\rho(\mathbf{r})$, then

$$\iint_S \mathbf{E}\cdot\mathbf{n}\,\mathrm{d}S = 4\pi \iiint_V \rho(\mathbf{r})\,\mathrm{d}^3\mathbf{r}, \tag{6.45}$$

where V is the volume enclosed by the surface S.

Now, by the theorem of Gauss (see Appendix A (A.36)), the surface integral is equal to the volume integral of the divergence of E,

$$\iint_S \mathbf{E}\cdot\mathbf{n}\,\mathrm{d}S = \iiint_V \nabla\cdot\mathbf{E}\,\mathrm{d}^3\mathbf{r}.$$

Hence (6.45) may be written

$$\iiint_V (\nabla\cdot\mathbf{E} - 4\pi\rho)\,\mathrm{d}^3\mathbf{r} = 0.$$

This equation must be true for an arbitrary volume V, and therefore the integrand itself must vanish,*

$$\nabla\cdot\mathbf{E} = 4\pi\rho. \tag{6.46}$$

The electric field is in fact completely determined (if we exclude the possibility of a uniform field extending over the whole of space)

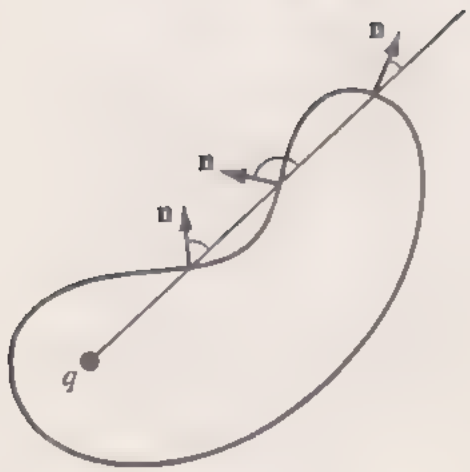


Fig. 6.8

* This equation takes a simpler form in SI units: $\nabla\cdot\mathbf{E} = \rho/\varepsilon_0$.

by this equation together with the condition that it be a conservative field,

$$\nabla \wedge \mathbf{E} = 0. \tag{6.47}$$

This latter equation guarantees the existence of the potential ϕ, related to $\mathbf{E}$ by $\mathbf{E} = -\nabla\phi$. (See (6.6).) Substituting in (6.46), we obtain *Poisson's equation* for the potential,

$$\nabla^2\phi = -4\pi\rho. \tag{6.48}$$

The main importance of these equations lies in the fact that they provide *local* relations between the potential or field and the charge density. The expression (6.15) for ϕ in terms of ρ is non-local, in the sense that it expresses the potential ϕ at one point as an integral involving the charge density at all points of space. To use this equation, we must know ρ everywhere. Frequently, however, we are interested in the field in some restricted region of space, where we do know the value of ρ, while, instead of having information about ρ outside this region, we have some conditions on the fields at the boundary. This type of problem may be solved by looking for solutions of (6.48) with appropriate boundary conditions. A particularly important case is that in which ρ vanishes inside the region of interest. Then (6.48) becomes *Laplace's* equation,

$$\nabla^2\phi = 0. \tag{6.49}$$

The gravitational case is of course entirely similar. The field equations are

$$\nabla\cdot\mathbf{g} = -4\pi G\rho, \tag{6.50}$$

where ρ is now the mass density, and

$$\nabla \wedge \mathbf{g} = 0. \tag{6.51}$$

The equation for the potential is

$$\nabla^2\Phi = 4\pi G\rho. \tag{6.52}$$

6.8 Summary

The great advantage of calculating the potential, rather than the field itself directly, is that it is much easier to add scalar quantities than vector ones. Even in simple cases, like that of the uniformly charged sphere, a direct integration of the forces produced by each element of the sphere is difficult to perform. In fact, the introduction of the potential was one of the major advances in mechanics, and

allowed many previously intractable problems to be handled relatively simply.

This chapter, in which we have discussed the method of determining the force on a particle from a knowledge of the positions of other masses or charges, is complementary to the previous chapters, where we discussed the motion of a particle under given forces. Together, they provide a method of solving most of the problems in which the object of immediate interest is (or may be taken to be) a single particle.

PROBLEMS

1 Find the potential and electric field at points on the axis of a uniformly charged flat circular disc of charge q and radius a. What happens to the field if we keep the charge density fixed, and let $a \to \infty$?

Calculate the quadrupole moment of the disc, and deduce the form of the field at large distances in any direction. Verify that on the axis it agrees with the exact value when r/a is sufficiently large.

2 Write down the potential energy of a pair of charges, q at $\mathbf{a}$ and $-q$ at the origin, in a field with potential $\phi(\mathbf{r})$. By considering the limit $a \to 0$ show that the potential energy of a dipole of moment $\mathbf{d}$ is $V = -\mathbf{d}\cdot\mathbf{E}$. If the electric field is uniform, when is this potential energy a minimum?

Show that the dipole experiences a couple $\mathbf{G} = \mathbf{d} \wedge \mathbf{E}$, and that in a non-uniform field there is also a net force $\mathbf{F} = (\mathbf{d}\cdot\nabla)\mathbf{E}$. (Take $\mathbf{d}$ in the z direction, and show that $\mathbf{F} = d\,\partial\mathbf{E}/\partial z$.)

3 Show that the work done in bringing two charges q_1 and q_2 from infinity to a separation r_{12} is $q_1 q_2/r_{12}$. Write down the corresponding expression for a system of many charges, noting that each pair must be counted only once. Hence show that the energy stored in the charge distribution is

$$V = \tfrac{1}{2} \sum q_i \phi(\mathbf{r}_i).$$

Why does this factor of $\frac{1}{2}$ appear here, but not if ϕ represents the potential of an *external* field? Find the analogous result for a uniformly charged sphere of charge q and radius a, and show that infinite energy is required to compress the sphere to a point.

4 Write the electric field of a dipole in vector notation. Using the result of Problem 2, find the potential energy of a dipole of moment $\mathbf{d}$ in the field of another of moment $\mathbf{d}'$. Find the forces and couples acting between the dipoles if they are placed on the z-axis and (i) both are pointing in the z direction, (ii) both are pointing in the x direction, (iii) one is pointing in the z direction, and one in the x direction, and (iv) one is pointing in the x direction and one in the y direction.

5 By considering the equilibrium of a small volume element, show that in a fluid in equilibrium under pressure and gravitational forces, $\nabla p = \rho\mathbf{g}$, where ρ is the density. Deduce that for an incompressible fluid $p + \rho\Phi$ is a constant. Use this result to obtain a rough estimate of the pressure at the centre of the earth in (metric) tons weight per square centimetre. (Mean density of earth $= 5{\cdot}5$ g cm^{-3}.)

6 A diffuse spherical cloud of gas of density ρ is initially at rest, and starts to collapse under its own gravitational attraction. Find the radial velocity

of a particle which starts at a distance a from the centre when it reaches the distance r. Hence, neglecting other forces, show that every particle will reach the centre at the same instant, and that the time taken is $(3\pi/32\rho G)^{1/2}$, independent of the initial radius of the cloud. Evaluate this time in years if $\rho = 10^{-22}$ g cm^{-3}.

7 If the mass of this gas cloud is 10^{27} t, and if the contraction is halted by the build-up of pressure when a star of radius 10^6 km has been formed, find the total energy released, assuming that the density of the star is uniform.

8 The rotation period of Jupiter is approximately 10 hours. Its mass and radius are 320 M_E and 11 R_E respectively (E = earth). Calculate approximately its oblateness, neglecting the variation of density. (The observed value is about 1/15.)

9 Assume that the earth consists of a core of uniform density ρ_1, surrounded by a mantle of uniform density ρ_2, and that the boundary between the two is of similar shape to the outer surface, but only three-fifths as large. Find what ratio of densities would be required to explain the observed quadrupole moment. (Treat the earth as a superposition of ellipsoids of densities ρ_2 and $\rho_1 - \rho_2$.)

10 Show that the moment of the earth's gravitational force may be written in the form

$$m\mathbf{r} \wedge \mathbf{g} = \frac{6GMma^2\lambda}{5r^5}(\mathbf{k}\cdot\mathbf{r})(\mathbf{k} \wedge \mathbf{r}).$$

Consider a satellite in a circular orbit of radius r in a plane inclined to the equator at an angle α. By introducing a pair of axes in the plane of the orbit, as in §5.6, show that the average value of this moment is

$$\langle m\mathbf{r} \wedge \mathbf{g}\rangle_{\text{av.}} = -\frac{3GMma^2\lambda}{5r^3}\cos\alpha(\mathbf{k} \wedge \mathbf{n}),$$

where $\mathbf{n}$ is the normal to the orbital plane. Hence show that the orbit precesses around the direction of the earth's axis, $\mathbf{k}$, at a rate

$$\Omega = -\frac{3\lambda}{5}\frac{a^2}{r^2}\omega\cos\alpha.$$

Evaluate this rate for an orbit 400 km above the earth's surface, with an inclination of 30°.

11 Verify that the potential of a uniformly charged sphere satisfies Poisson's equation, both inside and outside the sphere.

12 Assume that the pressure p in a star with spherical symmetry is related to the density ρ by the (somewhat unrealistic) equation of state $p = \frac{1}{2}k\rho^2$, where k is a constant. Use the fluid equilibrium equation obtained in Problem 5 to find a relation between ρ and Φ. Hence show that Poisson's equation yields

$$\frac{d^2[r\rho(r)]}{dr^2} = -\frac{4\pi G}{k}r\rho(r).$$

(You will need (A.51).) Solve this equation with the boundary conditions that ρ is finite at $r = 0$, and vanishes at the surface of the star. Hence show that the radius a of the star is determined solely by k, and is independent of its mass M. Show also that $M = (4/\pi)a^3\rho(0)$.

Chapter 7 The Two-Body Problem

We shall be mainly concerned in this chapter with an isolated system of two particles, subject only to the force between the two. However, as it is no harder to solve, and considerably extends the range of applicability of the results, we shall also allow the presence of a *uniform* gravitational field.

7.1 Centre-of-Mass and Relative Co-ordinates

We denote the positions and masses of the two particles by $\mathbf{r}_1$, $\mathbf{r}_2$ and m_1, m_2. If the force on the first particle due to the second is $\mathbf{F}$, then, by Newton's third law, that on the second due to the first is $-\mathbf{F}$. Thus, in a uniform gravitational field $\mathbf{g}$ the equations of motion are

$$\begin{aligned} m_1\ddot{\mathbf{r}}_1 &= m_1\mathbf{g}+\mathbf{F}, \\ m_2\ddot{\mathbf{r}}_2 &= m_2\mathbf{g}-\mathbf{F}. \end{aligned} \tag{7.1}$$

It is convenient to introduce new variables in place of $\mathbf{r}_1$ and $\mathbf{r}_2$. We define the position of the *centre of mass**

$$\mathbf{R} = \frac{m_1\mathbf{r}_1+m_2\mathbf{r}_2}{m_1+m_2}, \tag{7.2}$$

and the *relative* position

$$\mathbf{r} = \mathbf{r}_1-\mathbf{r}_2. \tag{7.3}$$

(See Fig. 7.1.) Differentiating twice, we find for $\mathbf{R}$ and $\mathbf{r}$ the equations of motion

$$M\ddot{\mathbf{R}} = M\mathbf{g}, \qquad M = m_1+m_2, \tag{7.4}$$

$$\mu\ddot{\mathbf{r}} = \mathbf{F}, \qquad \mu = \frac{m_1 m_2}{m_1+m_2}. \tag{7.5}$$

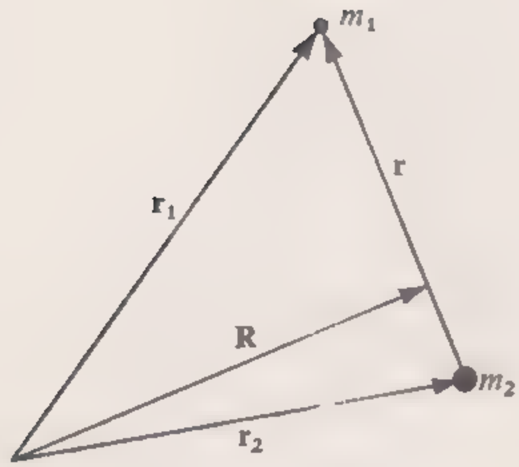


Fig. 7.1

These two equations are now completely separate. Equation (7.4) shows that the centre of mass moves with uniform acceleration $\mathbf{g}$. In the case $\mathbf{g} = \mathbf{0}$, it is equivalent to the law of conservation of momentum,

$$M\dot{\mathbf{R}} = m_1\dot{\mathbf{r}}_1+m_2\dot{\mathbf{r}}_2 = \mathbf{P} = \text{constant}. \tag{7.6}$$

* The centre of mass is more often denoted by $\bar{\mathbf{r}}$. This symbol is, however, typographically inconvenient when dots are used to denote time derivatives. We shall always use the symbol $\mathbf{R}$, which serves to emphasize the fact (established below) that the centre of mass is associated with the total mass M.

The equation of motion (7.5) for the relative position is identical with the equation for a single particle of mass μ moving under the force $\mathbf{F}$. The mass μ is called the *reduced mass* (because it is smaller than either m_1 or m_2). If we can solve this one-particle problem, then we can also solve the two-particle problem. When we have found $\mathbf{R}$ and $\mathbf{r}$ as functions of time, we can obtain the positions of the particles by solving the simultaneous equations (7.2) and (7.3):

$$\mathbf{r}_1 = \mathbf{R}+\frac{m_2}{M}\mathbf{r}, \qquad \mathbf{r}_2 = \mathbf{R}-\frac{m_1}{M}\mathbf{r}. \tag{7.7}$$

The separation between centre of mass and relative motion extends to the expressions for the total angular momentum and kinetic energy. We have

$$\begin{aligned}\mathbf{J} &= m_1\mathbf{r}_1\wedge\dot{\mathbf{r}}_1+m_2\mathbf{r}_2\wedge\dot{\mathbf{r}}_2\\ &= m_1\left(\mathbf{R}+\frac{m_2}{M}\mathbf{r}\right)\wedge\left(\dot{\mathbf{R}}+\frac{m_2}{M}\dot{\mathbf{r}}\right)+m_2\left(\mathbf{R}-\frac{m_1}{M}\mathbf{r}\right)\wedge\left(\dot{\mathbf{R}}-\frac{m_1}{M}\dot{\mathbf{r}}\right),\end{aligned}$$

by (7.7). It is easy to see that the cross terms between $\mathbf{R}$ and $\mathbf{r}$ cancel, so that we are left with

$$\mathbf{J} = M\mathbf{R}\wedge\dot{\mathbf{R}}+\mu\mathbf{r}\wedge\dot{\mathbf{r}}. \tag{7.8}$$

Similarly, substituting (7.7) into

$$T = \tfrac{1}{2}m_1\dot{\mathbf{r}}_1{}^2+\tfrac{1}{2}m_2\dot{\mathbf{r}}_2{}^2,$$

we find

$$T = \tfrac{1}{2}M\dot{\mathbf{R}}^2+\tfrac{1}{2}\mu\dot{\mathbf{r}}^2. \tag{7.9}$$

It is instructive to examine this problem also from the Lagrangian point of view introduced in §3.6. Let us suppose that the force $\mathbf{F}$ is conservative, and corresponds to a potential energy function $V_{\text{int}}(\mathbf{r})$. The total potential energy V includes not only V_{int} but also the potential energy of the external gravitational forces. Now the potential energy of a particle of mass m in a uniform gravitational field $\mathbf{g}$ is $-m\mathbf{g}\cdot\mathbf{r}$. (If $\mathbf{g}$ is in the $-z$ direction, this is the familiar expression mgz.) Thus the Lagrangian function is ← from §8.6 Lagrange's Equations p134

$$\begin{aligned}L = T-V = {}&\tfrac{1}{2}m_1\dot{\mathbf{r}}_1{}^2+\tfrac{1}{2}m_2\dot{\mathbf{r}}_2{}^2\\ &+m_1\mathbf{g}\cdot\mathbf{r}_1+m_2\mathbf{g}\cdot\mathbf{r}_2-V_{\text{int}}(\mathbf{r}_1-\mathbf{r}_2).\end{aligned} \tag{7.10}$$

It is easy to verify that Lagrange's equations yield the correct equations of motion. Since

$$\frac{\partial V_{\text{int}}}{\partial x_1} = \frac{\partial V_{\text{int}}}{\partial x} = -F_x, \qquad \frac{\partial V_{\text{int}}}{\partial x_2} = -\frac{\partial V_{\text{int}}}{\partial x} = F_x,$$

we have

$$\frac{\partial L}{\partial \dot{x}_1} = m_1\dot{x}_1, \qquad \frac{\partial L}{\partial x_1} = m_1 g_x + F_x,$$

$$\frac{\partial L}{\partial \dot{x}_2} = m_2\dot{x}_2, \qquad \frac{\partial L}{\partial x_2} = m_2 g_x - F_x.$$

Hence Lagrange's equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = \frac{\partial L}{\partial x_1}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_2}\right) = \frac{\partial L}{\partial x_2},$$

yield precisely the x components of (7.1).

Now, just as in §3.6, we may express L in terms of any six independent co-ordinates in place of $x_1, y_1, z_1, x_2, y_2, z_2$. In particular we may choose the co-ordinates of $\mathbf{R}$ and $\mathbf{r}$. Thus from (7.9) and (7.10), we find

$$L = \tfrac{1}{2}M\dot{\mathbf{R}}^2 + M\mathbf{g}\cdot\mathbf{R} + \tfrac{1}{2}\mu\dot{\mathbf{r}}^2 - V_{\mathrm{int}}(\mathbf{r}). \tag{7.11}$$

The complete separation of the terms involving $\mathbf{R}$ from those involving $\mathbf{r}$ is now obvious. It is easy to check that Lagrange's equations for these co-ordinates are just (7.4) and (7.5).

In this form, it is clear that the crucial point is the separation of the *potential* energy. The kinetic energy may always be separated according to (7.9), but only for a *uniform* gravitational field does the potential energy separate in a similar way.

7.2 The Centre-of-Mass Frame

It is often convenient to describe the motion of the system in terms of a frame of reference in which the centre of mass is at rest at the origin. (In a gravitational field, this is an accelerated, non-inertial frame, but it is still useful.) This is the *centre-of-mass* (*CM*) *frame*. We shall denote quantities referred to it by an asterisk.

The relative position $\mathbf{r}$ is of course independent of the choice of origin, so that setting $\mathbf{R}^* = \mathbf{0}$ in (7.7) we find

$$\mathbf{r}_1^* = \frac{m_2}{M}\mathbf{r}, \qquad \mathbf{r}_2^* = -\frac{m_1}{M}\mathbf{r}. \tag{7.12}$$

In this frame, the momenta of the two particles are equal and opposite,

$$m_1\dot{\mathbf{r}}_1^* = -m_2\dot{\mathbf{r}}_2^* = \mu\dot{\mathbf{r}} = \mathbf{p}^*, \tag{7.13}$$

say.

As we shall see explicitly later, it is often convenient to solve a problem first in the CM frame. To find the solution in some other frame, we then need the relations between the momenta in the two frames. Let us consider a frame in which the centre of mass is moving with velocity $\dot{\mathbf{R}}$. Then the velocities of the two particles are

$$\dot{\mathbf{r}}_1 = \dot{\mathbf{R}}+\dot{\mathbf{r}}_1^*, \qquad \dot{\mathbf{r}}_2 = \dot{\mathbf{R}}+\dot{\mathbf{r}}_2^*.$$

Hence by (7.13) their momenta are

$$\mathbf{p}_1 = m_1\dot{\mathbf{R}}+\mathbf{p}^*, \qquad \mathbf{p}_2 = m_2\dot{\mathbf{R}}-\mathbf{p}^*. \tag{7.14}$$

From (7.8) and (7.9) it follows that the total angular momentum and kinetic energy in the CM frame are

$$\begin{aligned} \mathbf{J}^* &= \mu\mathbf{r}\wedge\dot{\mathbf{r}} = \mathbf{r}\wedge\mathbf{p}^*, \\ T^* &= \tfrac{1}{2}\mu\dot{\mathbf{r}}^2 = \frac{\mathbf{p}^{*2}}{2\mu}. \end{aligned} \tag{7.15}$$

Thus in any other frame we can write

$$\begin{aligned} \mathbf{P} &= M\dot{\mathbf{R}}, \\ \mathbf{J} &= M\mathbf{R}\wedge\dot{\mathbf{R}}+\mathbf{J}^*, \\ T &= \tfrac{1}{2}M\dot{\mathbf{R}}^2+T^*. \end{aligned} \tag{7.16}$$

To obtain the values in any frame from those in the CM frame, we have only to add the contribution of a particle of mass M located at the centre of mass $\mathbf{R}$. We shall see in the next chapter that this is a general conclusion, not restricted to two-particle systems.

Motion of a Satellite. As an example, let us consider two bodies moving under their mutual gravitational attraction. Equation (7.5) becomes

$$\mu\ddot{\mathbf{r}} = -\hat{\mathbf{r}}\frac{Gm_1m_2}{r^2} = -\hat{\mathbf{r}}\frac{GM\mu}{r^2}.$$

This is identical with the equation for a particle moving around a fixed mass M. In particular, for an elliptic orbit, the period is given by (4.34),

$$\left(\frac{\tau}{2\pi}\right)^2 = \frac{a^3}{GM}.$$

Note that a here is the semi-major axis of the *relative* orbit (the median distance between the bodies), and that M is the sum of the masses,

rather than the mass of the heavier body. Thus Kepler's third law is only approximately correct: the orbital period depends not only on the semi-major axis, but also on the mass.

The only case in the solar system for which the lighter mass is an appreciable fraction of the total is that of the earth–moon system, for which $m_1/m_2 = 1/81{\cdot}5$. If we were to compute the period of the moon's orbit from Kepler's third law by comparing it with the period of a small earth satellite, we should obtain the value given above but with m_2 in place of M. This would yield a period about 4 hours too long.

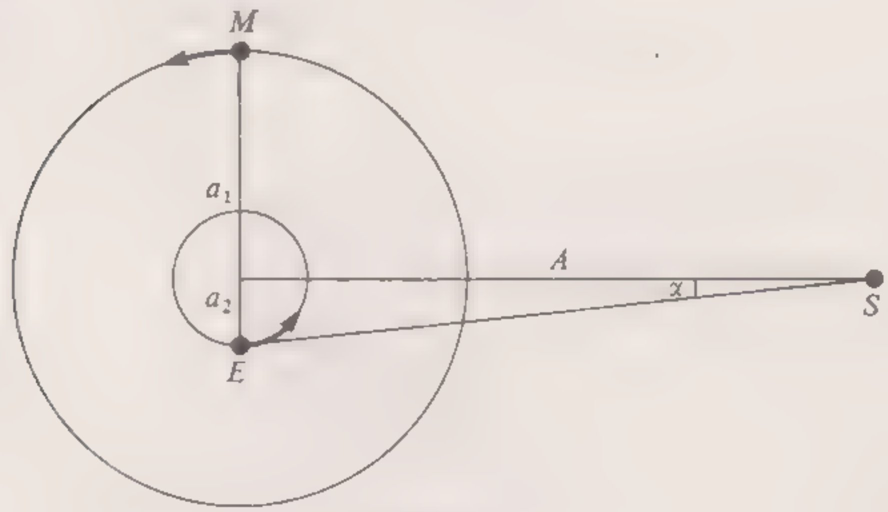


Fig. 7.2

In the CM frame of the earth–moon system, we see from (7.12) that both bodies move in ellipses around the centre of mass, with semi-major axes

$$a_1 = \frac{m_2}{M}a, \qquad a_2 = \frac{m_1}{M}a.$$

For the earth–moon system, $a = 3{\cdot}8 \times 10^5$ km. The earth therefore moves around the centre of mass in a small ellipse with semi-major axis $a_2 = a/82{\cdot}5 = 4700$ km. This leads to a small oscillation in the apparent direction of the sun (see Fig. 7.2) of angular amplitude $\alpha \approx a_2/A$, where A is the distance to the sun. Since $A = 1{\cdot}5 \times 10^8$ km, $\alpha = 6{\cdot}5''$. A larger effect can be obtained by looking at the direction of some object closer to us than the sun, such as an asteroid. This effect provides a method of determining the lunar mass.

In the frame in which the sun is at rest, the centre of mass of the earth–moon system moves in an ellipse around the sun, and the motion of the two bodies around the centre of mass is superimposed

on this larger orbital motion. In the approximation in which the sun's gravitational field is uniform over the dimensions of the earth–moon system, the two types of motion are completely separate, and no transfer of energy or angular momentum from one to the other can occur. However, as we shall see later, there are small effects due to the non-uniformity of the sun's field.

7.3 Elastic Collisions

A collision between two particles is called *elastic* if there is no loss of kinetic energy in the collision; that is, if the total kinetic energy after the collision is the same as that before it. Such collisions are

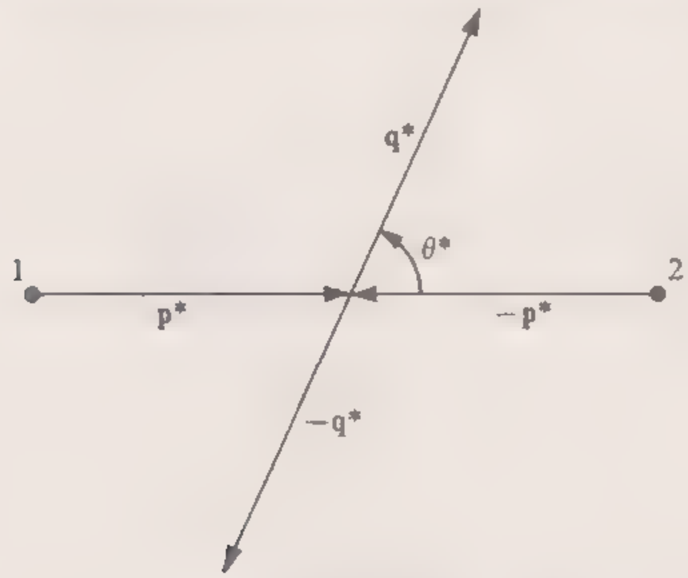


Fig. 7.3

typical of very hard bodies, like billiard balls. (They are also important in atomic and nuclear collision problems.)

It is easy to describe such a collision in the CM frame. The particles must approach each other with equal and opposite momenta, $\mathbf{p}^*$ and $-\mathbf{p}^*$, and recede after the collision again with equal and opposite momenta, $\mathbf{q}^*$ and $-\mathbf{q}^*$. Thus each particle is scattered through the same angle θ^*. (See Fig. 7.3.) Since the collision is elastic, we have, from (7.15),

$$T^* = \frac{\mathbf{p}^{*2}}{2\mu} = \frac{\mathbf{q}^{*2}}{2\mu}. \tag{7.17}$$

Thus the magnitudes of the momenta before and after the collision are the same,

$$p^* = q^*. \tag{7.18}$$

In practice, most experiments are performed with one particle initially at rest (or nearly so) in the laboratory. To interpret such an experiment, we therefore need to use the *laboratory* (*Lab*) *frame*, in which the momentum of particle 2 before the collision is zero, $\mathbf{p}_2 = \mathbf{0}$.

We shall denote the Lab momentum of the incoming particle by $\mathbf{p}_1$, the momenta after the collision by $\mathbf{q}_1$ and $\mathbf{q}_2$, and the angles of scattering and recoil by θ and α. (See Fig. 7.4.) We could work out the relations between these quantities by using the conservation laws of momentum and energy directly in the Lab frame, but it is actually simpler to relate them to the CM quantities, using the formulae obtained in §7.2.

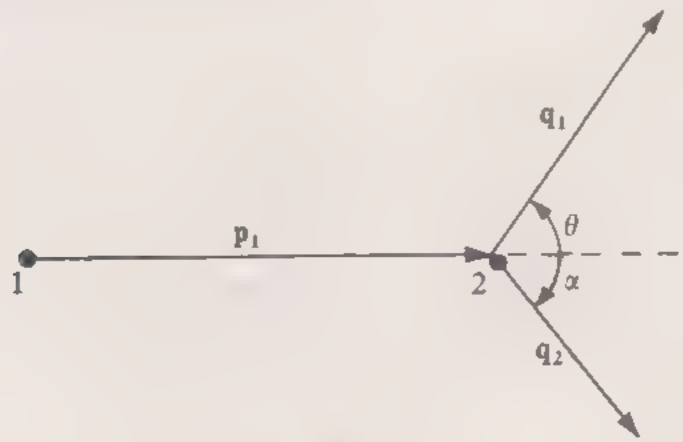


Fig. 7.4

Since $\mathbf{p}_2 = \mathbf{0}$, we have from (7.14)

$$\dot{\mathbf{R}} = \frac{1}{m_2}\mathbf{p}^*, \tag{7.19}$$

and also

$$\mathbf{p}_1 = \frac{m_1}{m_2}\mathbf{p}^* + \mathbf{p}^* = \frac{M}{m_2}\mathbf{p}^*. \tag{7.20}$$

The momenta after the collision are again given by (7.14) but with $\mathbf{q}^*$ in place of $\mathbf{p}^*$. They are

$$\mathbf{q}_1 = \frac{m_1}{m_2}\mathbf{p}^* + \mathbf{q}^*, \qquad \mathbf{q}_2 = \mathbf{p}^* - \mathbf{q}^*.$$

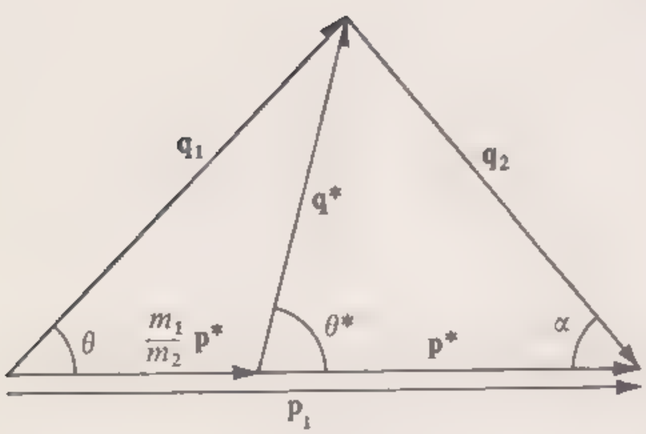


Fig. 7.5

All these relations may be conveniently summarized in a vector diagram, drawn in Fig. 7.5, which also incorporates the momentum conservation equation in the Lab, $\mathbf{p}_1 = \mathbf{q}_1 + \mathbf{q}_2$. Any desired relation between momenta, energies or angles may be extracted from this

diagram without difficulty. We consider explicitly only a few important relations.

By (7.18), the vectors $\mathbf{p}^*$, $\mathbf{q}^*$, $\mathbf{q}_2$ form an isosceles triangle. Thus the recoil angle α and the recoil momentum $\mathbf{q}_2$ are given in terms of CM quantities by

$$\alpha = \tfrac{1}{2}(\pi-\theta^*),$$
$$q_2 = 2p^* \sin \tfrac{1}{2}\theta^*. \tag{7.21}$$

The Lab kinetic energy transferred to the target particle is therefore

$$T_2 = \frac{q_2{}^2}{2m_2} = \frac{2p^{*2}}{m_2} \sin^2 \tfrac{1}{2}\theta^*.$$

On the other hand, the total kinetic energy in the Lab is just the kinetic energy of the incoming particle,

$$T = \frac{p_1{}^2}{2m_1} = \frac{M^2 p^{*2}}{2m_1 m_2{}^2},$$

by (7.20). The interesting quantity is the fraction of the total kinetic energy which is transferred. This is

$$\frac{T_2}{T} = \frac{4m_1 m_2}{M^2} \sin^2 \tfrac{1}{2}\theta^*. \tag{7.22}$$

The maximum possible kinetic energy transfer occurs for a head-on collision ($\theta^* = \pi$), and is $4m_1m_2/(m_1+m_2)^2$. Clearly, this can be close to unity only if m_1 and m_2 are comparable in magnitude. If the incoming particle is very light, it bounces off the target with little loss of energy; if it is very heavy, it is hardly deflected at all from its original trajectory, and again loses little of its energy. For example, in a proton–α-particle collision ($m_1/m_2 = 4$ or $\frac{1}{4}$), the maximum fractional energy transfer is 64%; in a proton–electron collision ($m_1/m_2 = 1836$) it is about 0·2%.

Another important relation is that between the Lab and CM scattering angles. It is easy to prove by elementary trigonometry (most simply by dropping a perpendicular from the upper vertex of Fig. 7.5) that quoted for (7.31)

$$\tan \theta = \frac{\sin \theta^*}{(m_1/m_2)+\cos \theta^*}. \tag{7.23}$$

Note that this relation is independent of the momenta of the two particles, and depends only on their mass ratio.

As θ^* varies from 0 to π, the vector $\mathbf{q}^*$ sweeps out a semicircle of radius p^*. If the target is the heavier particle ($m_1/m_2<1$), the left-hand vertex of Fig. 7.5 lies inside this semicircle, and θ also varies

from 0 to π. However, if the target is the lighter particle ($m_1/m_2>1$), the semicircle excludes this vertex. In that case, $\theta = 0$ when θ^* is either 0 or π, and there is an intermediate value of θ^* for which θ is a maximum. This maximum scattering angle occurs when $\mathbf{q}_1$ is tangent to the circle (see Fig. 7.6). It is given by

$$\sin\theta_{\max} = m_2/m_1. \tag{7.24}$$

For instance, an α-particle can only be scattered by a proton through an angle less than 14·5°, and a proton can only be scattered by an electron through an angle less than 0·03°.

In the special case of equal-mass particles, the right side of (7.23) simplifies to $\tan\frac{1}{2}\theta^*$, so that we obtain

$$\theta = \tfrac{1}{2}\theta^*, \qquad (m_1 = m_2). \tag{7.25}$$

In this case, the maximum scattering angle is $\frac{1}{2}\pi$.

7.4 CM and Lab Cross-Sections

Let us now consider the scattering of a beam of particles by a target. We shall suppose that the target contains a large number N of identical target particles, but is still small enough to have negligible size. We also suppose that the beam is a parallel beam of particles, each with the same mass and velocity, with a particle flux of f particles crossing unit area per unit time. The incident particles may or may not be the same as those in the target, but for the moment we shall assume that they can be distinguished, so that it is possible to set up a detector to count the number of scattered particles emerging in some direction, without including also the recoiling target particles.

We can now introduce the concept of differential cross-section, just as we did for the case of fixed target particles in §4.5. Indeed the fact that the particles which are struck recoil out of the target makes no essential difference. In any one collision, the scattering angle θ will be determined in some way (depending on the shape of the particles) by the impact parameter b. Thus the particles scattered through angles between θ and $\theta+\mathrm{d}\theta$ will be those of the incident particles which strike any one of the target particles with impact parameters between the corresponding values b and $b+\mathrm{d}b$. To find the number emerging within a solid angle $\mathrm{d}\Omega$ in some specified direction, we have to calculate the corresponding cross-sectional area of the incident beam,

$$\mathrm{d}\sigma = b|\mathrm{d}b|\mathrm{d}\varphi, \tag{7.26}$$

(compare (4.41) and Fig. 4.8), and multiply by the number of target particles, N, and by the flux, f.

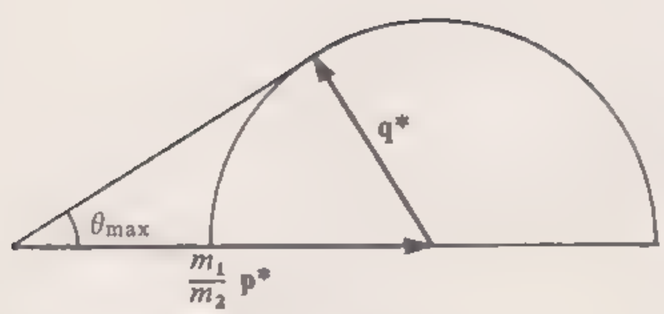


Fig. 7.6

If we set up a detector of cross-sectional area $\mathrm{d}A$ at a large distance from the target, the rate of detection will be

$$\mathrm{d}w = Nf\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\frac{\mathrm{d}A}{L^2}, \tag{7.27}$$

exactly as in (4.49). The ratio $\mathrm{d}\sigma/\mathrm{d}\Omega$ is the *Lab differential cross-section.*

We shall return to the problem of calculating the Lab differential cross-section later. First, however, we wish to discuss a slightly different type of experiment.

Let us imagine two beams of particles approaching from opposite directions. In particular, we shall be interested in the case where the momenta of the particles in the beams are equal and opposite, so that we are directly concerned with the CM frame. To be specific, let us suppose that the particles in one beam are hard spheres of radius a_1 and those in the other are hard spheres of radius a_2. Evidently, a particular pair of particles will collide if the distance b between the lines of motion of their centres is less than $a = a_1 + a_2$. (See Fig. 7.7.) Let us select one of the particles in the second beam. It will collide with any particles in the first beam whose centres cross an area $\sigma = \pi a^2$. This is the *total cross-section* for the collision, and is the same whether the target is at rest or moving towards the beam. Note that it is the *sum* of the radii which enters here. This serves to emphasize the fact that the cross-section is a property of the pair of particles involved in the collision, not of either particle individually.

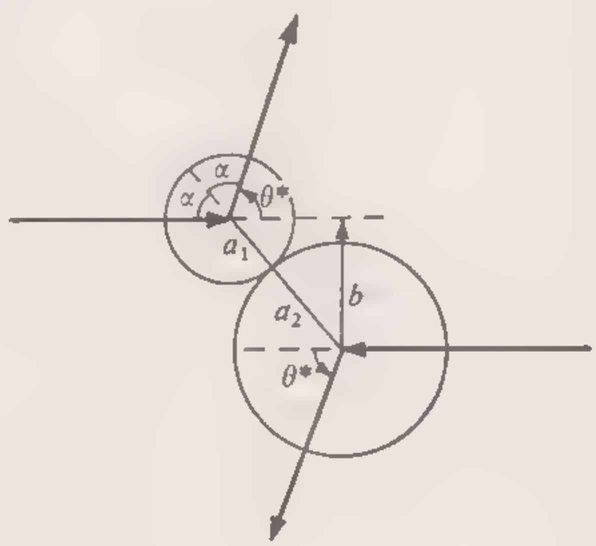


Fig. 7.7

To compute the number of collisions occurring in a short time interval $\mathrm{d}t$, we may imagine the cross-sectional area σ to be attached to our selected particle, and moving with it. Then the probability that the centre of one of the particles in the first beam crosses this area in the time interval $\mathrm{d}t$ will be $n_1 v\sigma\,\mathrm{d}t$, where n_1 is the number of particles per unit volume in the beam, and v is the relative velocity $(= v_1 + v_2)$. If the number of particles in the second beam per unit volume is n_2, then the total number of collisions occurring in volume V is $n_1 n_2 v\sigma V\,\mathrm{d}t$.

Now, if we are interested in the number of particles emerging in a given solid angle, we have to find the relation between b and the CM scattering angle θ^*, and evaluate the corresponding cross-sectional area (7.26). In our case, it is clear from Fig. 7.7 that

$$b = a\sin\alpha = a\cos\tfrac{1}{2}\theta^*. \tag{7.28}$$

Hence, introducing the solid angle

$$\mathrm{d}\Omega^* = \sin\theta^*\,\mathrm{d}\theta^*\,\mathrm{d}\varphi,$$

we find that the *CM differential cross-section* is

$$\frac{d\sigma}{d\Omega^*} = \tfrac{1}{4}a^2. \tag{7.29}$$

If we set up a detector arranged to record particles scattered from a small volume V, then the rate of detection of particles will be

$$dw = n_1 n_2 v V \frac{d\sigma}{d\Omega^*}\frac{dA}{L^2}. \tag{7.30}$$

Note that N and f in (7.27) correspond to $n_2 V$ and $n_1 v$ in this formula.

In our particular case, the differential cross-section (7.29) is independent of θ^*, and the scattering is therefore isotropic in the CM frame. In other words, the number of particles detected is independent of the direction in which we place the detector. This is a very special property of hard spheres, and is not true in general. Nor is it true in the Lab frame. Indeed, the reason for calculating the CM differential cross-section first is that the relation between b and θ^* is much simpler than that between b and θ, so that the cross-section takes a simpler form in the CM frame than in the Lab frame.

The easiest way to find the Lab differential cross-section is often, to compute the CM cross-section first, and then use the relation

$$\frac{\delta\Omega^*}{\delta\Omega^*} = \frac{\sin\theta^*\,\delta\theta}{\sin\theta\,\delta\theta} \to \frac{\delta z^*}{\delta z}$$

? $$\frac{d\Omega^*}{d\Omega} = \frac{\sin\theta^*\,d\theta^*}{\sin\theta\,d\theta} = \frac{dz^*}{dz},$$

where $z = \cos\theta$ and $z^* = \cos\theta^*$. Let us assume (for a reason to be given later) that $m_1 < m_2$. Then it follows that

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{d\Omega^*}\frac{dz^*}{dz}. \tag{7.31}$$

The value of dz^* dz may be found from (7.23), which can be written

$$z^2 = \frac{[(m_1/m_2)+z^*]^2}{1+2(m_1, m_2)z^* + (m_1/m_2)^2}.$$

It is straightforward, if tedious, to solve for z^* and differentiate, but the resulting expression for the general case is not particularly illuminating, and we shall omit it.

If $m_1 > m_2$, there is a complication which arises from the fact that for each value of θ less than the maximum scattering angle θ_{max}, there are two possible values of θ^*, as is clear from Fig. 7.6. If we measure only the direction of the scattered particles, we cannot distinguish these two, and must add the contributions to $d\sigma/d\Omega$ from $d\sigma/d\Omega^*$ at two separate values of θ^*. (Moreover, for one of these values dz^*/dz

is negative, and must be replaced by its absolute value.) The two values of θ^* can be distinguished if we measure also the energy or momentum of the scattered particles.

We shall consider explicitly only the case of equal mass particles, $m_1 = m_2$. Then the relation between scattering angles simplifies to $\theta^* = 2\theta$, or $z^* = 2z^2-1$. Consequently, $\mathrm{d}z^*/\mathrm{d}z = 4z$, and the Lab differential cross-section is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = a^2 \cos\theta, \qquad \theta<\tfrac{1}{2}\pi. \tag{7.32}$$

Of course, no particles are scattered through angles greater than $\frac{1}{2}\pi$, so

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = 0, \qquad \theta>\tfrac{1}{2}\pi. \tag{7.33}$$

In this case, the cross-section is peaked towards the forward direction, and more particles will enter a detector placed at a small angle to the beam direction than one placed nearly at right angles.

As a check of the result (7.32), we may evaluate the total cross-section σ, which must agree with the value $\sigma = 4\pi\times\frac{1}{4}a^2 = \pi a^2$ obtained in the CM frame. We find

$$\sigma = \iint \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\,\mathrm{d}\Omega = \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\frac{1}{2}\pi} a^2 \cos\theta \sin\theta\,\mathrm{d}\theta = \pi a^2,$$

as expected. (Note the upper limit $\frac{1}{2}\pi$ on the θ integration, which occurs because of (7.33).)

We assumed at the beginning of this section that the target particles and beam particles were distinguishable, but it is easy to relax this condition. We shall now calculate the rate at which recoiling target particles enter the detector. In a collision in which the incoming particle has impact parameter b, and is moving in a plane defined by the angle φ, the target particle will recoil in a direction specified by the polar angles α, ψ, where α is related to θ^* or b, and $\psi = \pi+\varphi$. The number of target particles emerging within an angular range $\mathrm{d}\alpha$, $\mathrm{d}\psi$, is therefore equal to the number of incoming particles in the corresponding range $\mathrm{d}b$, $\mathrm{d}\varphi$. To determine the number of recoiling particles entering the detector we have to relate $\mathrm{d}\sigma$ to the solid angle

$$\mathrm{d}\Omega_2 = \sin\alpha\,\mathrm{d}\alpha\,\mathrm{d}\psi.$$

By (7.21), we have $\cos\theta^* = 1-2\cos^2\alpha$, whence

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_2} = \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega^*}\,4\cos\alpha, \qquad \alpha<\tfrac{1}{2}\pi. \tag{7.34}$$

In the particular case of hard spheres, this differential cross-section has exactly the same form as (7.32). This shows that in the scattering of identical hard spheres the numbers of scattered particles and recoiling target particles entering the detector are precisely equal, and the total detection rate is obtained by simply doubling that found earlier.

7.5 Summary

For our later work, the most important result of this chapter is that the total momentum, angular momentum and kinetic energy in an arbitrary frame differ from those in the CM frame by an amount equal to the contribution of a particle of mass M moving with the centre of mass.

We have seen that for a two-particle system the use of the CM frame is often a considerable simplification, and this is true also for more complicated systems. When we need results in some other frame, it is often best to solve the problem first in the CM frame, and then transform to the required frame.

PROBLEMS

1 A double star is formed of two components, each with mass equal to that of the sun. The distance between them is the same as that between the earth and sun. What is the orbital period?

2 The parallax of a star (the angle subtended at the star by the radius of the earth's orbit) is p. The star's position is observed to oscillate with angular amplitude α, and period τ. If this oscillation is interpreted as being due to the existence of a planet moving in a circular orbit around the star, show that its mass m_1 is given by

$$\frac{m_1}{M_0} = \frac{\alpha}{p}\left(\frac{M\tau_0}{M_0\tau}\right)^{2/3},$$

where M is the total mass of star plus planet, M_0 is the sun's mass, and $\tau_0 = 1$ year. Evaluate the mass if $M = 0{\cdot}25\ M_0$, $\tau = 16$ years, $p = 0{\cdot}5''$ and $\alpha = 0{\cdot}01''$. What conclusion can be drawn without making the assumption that the orbit is a circle?

3 Two particles of masses m_1 and m_2 are attached to the ends of a light spring. The natural length of the spring is l, and its tension is k times its extension. Initially, the particles are at rest, with m_1 at a height l above m_2. At $t = 0$, m_1 is projected vertically upward with velocity v. Find the positions of the particles at any subsequent time. What is the largest value of v for which this solution applies?

4 Two identical charged particles, each of mass m and charge e, are initially far apart. One of the particles is at rest, and the other is moving with velocity v and impact parameter $b = 2e^2/mv^2$. Find the distance of

closest approach of the particles, and the velocity of each at the moment of closest approach.

5 Show that in an elastic scattering process the angle between the emerging particles is given by

$$\tan(\theta+\alpha) = \frac{m_1+m_2}{m_1-m_2}\cot\tfrac{1}{2}\theta^*.$$

What is the mass ratio if the particles emerge at right angles?

6 A proton is elastically scattered through an angle of 56° by a nucleus, which recoils at an angle of 60°. Find the atomic mass of the nucleus, and the fraction of the kinetic energy transferred to it.

7 Obtain the relation between the total kinetic energy in the CM and Lab frames. Discuss the limiting cases of very large and very small mass for the target.

8 An experiment is to be designed to measure the differential cross-section for elastic pion–proton scattering at a CM scattering angle of 70° and a pion CM kinetic energy of 490 keV. (The electron-volt (eV) is the atomic unit of energy.) Find the angles in the Lab at which the scattered pions, and the recoiling protons, should be detected, and the required Lab kinetic energy of the pion beam. (The ratio of pion to proton mass is 1/7.)

9 Calculate the differential cross-section for the scattering of identical hard spheres directly in the Lab frame.

10 Find the Lab differential cross-section for the scattering of identical particles of charge e and mass m, if the incident velocity is v.

11 At low energies, protons and neutrons behave roughly like hard spheres of radius about $2{\cdot}5\times10^{-12}$ cm. A parallel beam of neutrons, with a flux of 3×10^6 neutrons cm^{-2} s^{-1}, strikes a target containing 4×10^{22} protons. A circular detector of radius 2 cm is placed 70 cm from the target at an angle of 30° to the beam direction. Calculate the rate of detection of neutrons, and of protons.

12 An unstable particle of mass $M = m_1+m_2$ decays into two particles of masses m_1 and m_2, releasing an amount of energy Q. Determine the kinetic energies of the two particles in the CM frame. If the unstable particle is moving in the Lab with kinetic energy T, find the maximum and minimum kinetic energies of the particle of mass m_1 in the Lab.

13 The molecules in a gas may be treated as identical hard spheres. Find how many collisions are required, on average, to reduce the velocity of an exceptionally fast molecule by a factor of 1000. (Assume that its velocity is so large that the other molecules are effectively at rest.)

Chapter 8 Many-Body Systems

We now consider a general system of N particles labelled by an index $i = 1, 2, \ldots, N$, interacting through two-body forces and subjected also to external forces due to bodies outside the system. We denote the force on the ith particle due to the jth particle by $\mathbf{F}_{ij}$, and the external force on the ith particle by $\mathbf{F}_i$. Thus the equations of motion are

$$m_i\ddot{\mathbf{r}}_i = \sum_j \mathbf{F}_{ij}+\mathbf{F}_i. \tag{8.1}$$

Here, and throughout this chapter, the sum is over all particles of the system, $j = 1, 2, \ldots, N$. Of course, there is no force on the ith particle due to itself, and so $\mathbf{F}_{ii} = \mathbf{0}$. The sum in this case is really over the other $N-1$ particles.

8.1 Momentum; Centre-of-Mass Motion

The position $\mathbf{R}$ of the centre of mass is defined, as in (7.2), by

$$\mathbf{R} = \frac{1}{M}\sum_i m_i\mathbf{r}_i, \qquad M = \sum_i m_i. \tag{8.2}$$

The total momentum is

$$\mathbf{P} = \sum_i m_i\dot{\mathbf{r}}_i = M\dot{\mathbf{R}}. \tag{8.3}$$

It is equal to the momentum of a particle of mass M located at the centre of mass.

The rate of change of momentum, by (8.1), is

$$\dot{\mathbf{P}} = \sum_i\sum_j \mathbf{F}_{ij}+\sum_i \mathbf{F}_i. \tag{8.4}$$

Now the two-body forces $\mathbf{F}_{ij}$ must satisfy Newton's third law,

$$\mathbf{F}_{ji} = -\mathbf{F}_{ij}. \tag{8.5}$$

Thus, for every term $\mathbf{F}_{ij}$ in the double sum in (8.4), there is an equal and opposite term $\mathbf{F}_{ji}$. The terms therefore cancel in pairs, and the double sum is zero. (Compare the discussion at the end of §1.3 for the three-particle case.)

Hence we obtain the important result that the rate of change of momentum is equal to the sum of the *external* forces alone

$$\dot{\mathbf{P}} = M\ddot{\mathbf{R}} = \sum_i \mathbf{F}_i. \tag{8.6}$$

In the special case of an isolated system of particles, acted on by no external forces, this yields the law of *conservation of momentum*

$$\mathbf{P} = M\dot{\mathbf{R}} = \text{constant}. \tag{8.7}$$

In this case, the centre of mass moves with uniform velocity.

Let us now regard our system of particles as forming a composite body. If the body is isolated, then according to Newton's first law it moves with uniform velocity. Thus we see that to maintain this law for composite bodies, we should define the *position* of such a body to mean the position of its centre of mass. Moreover, with this definition, Newton's second law is just (8.6), provided that we interpret the *force* on the body in the obvious way to mean the sum of the forces on all its constituent particles, and the *mass* as the sum of their masses. It is also clear that if Newton's third law applies to each pair of particles from two composite bodies, then it will apply to the bodies as a whole.

Thus, with a suitable (and very natural) interpretation of the concepts involved, Newton's three basic laws may be applied to composite bodies as well as to point particles. (Though we have phrased the discussion in terms of collections of particles, we could include also the case of a continuous distribution of matter, by dividing it up into infinitesimal particles.) It follows that, so long as we are interested only in the motion of a body as a whole, we may replace it by a particle of mass M located at the centre of mass. This is a result of the greatest importance, for its allows us to apply our earlier discussion of particle motion to real physical bodies. We have of course implicitly assumed it in many of our applications: for example, we treated the planets as point particles in discussing their orbital motion.

There is one point about which one must be careful. We may still need some information about the actual shape of the body to calculate the total force acting on it. This force is not necessarily equal to that on a particle of mass M located at the centre of mass. In the gravitational case, it happens to be so if the body is spherical, or if the gravitational field is uniform, but not in general otherwise.

8.2 Rockets

As an example of the use of the momentum conservation equation for an isolated system, we consider the motion of a rocket.

We suppose that the rocket is emitting matter with a constant velocity u relative to the rocket, but not necessarily at a uniform rate.

Let the mass of the rocket at some instant be M, and its velocity v, and consider the emission of a small mass dm. After the emission, the mass of the rocket will be reduced by this amount,

$$dM = -dm, \tag{8.8}$$

and its velocity will be increased to some value $v+dv$. (See Fig. 8.1.)

Fig. 8.1

The momentum conservation equation reads

$$(M-dm)(v+dv)+dm(v-u) = Mv,$$

whence, neglecting second-order infinitesimals, we obtain

$$M\,dv = u\,dm. \tag{8.9}$$

By (8.8), we can write this in the form

$$\frac{dv}{u} = -\frac{dM}{M},$$

whence, integrating,

$$\frac{v}{u} = -\ln\frac{M}{M_0}, \tag{8.10}$$

where the constant of integration M_0 is the value of M when $v = 0$.

This relation can be solved to give the mass as a function of velocity,

$$M = M_0 e^{-v/u}. \tag{8.11}$$

This shows that to accelerate a rocket to a velocity equal to its ejection velocity u, we must eject all but a fraction $1/e$ of its original mass.

Note that the velocity of the rocket depends only on the ejection velocity and the fraction of the initial mass which has been ejected, and not on the rate of ejection. It makes no difference whether the acceleration is brief and intense, or prolonged and gentle. (This assumes of course that the rocket is not subjected to other forces during its acceleration. It would clearly be useless to try to escape from the earth with a rocket providing a long, slow acceleration, because it would be constantly retarded by gravity.)

For interplanetary flights, the rockets would normally be used only for brief spells, between which the spaceship moves in a free orbit. ? If the duration of the rocket bursts is sufficiently short for the change in position of the rocket during each burst to be negligible, then it is a good approximation to assume that on each occasion the velocity is changed instantaneously, by an amount known as the *velocity impulse*. The relevant quantity for determining the mass of rocket required to deliver a given payload, using a given ejection velocity, is then the sum (in the ordinary, not the vector, sense) of these velocity impulses. For example, the velocity impulse needed to escape from the earth is 11 km/s. For the return trip it is 22 km/s, if the deceleration on re-entry into the atmosphere is to be produced by the rockets rather than atmospheric friction.

8.3 Angular Momentum; Central Internal Forces

The total angular momentum of our system of particles is

$$\mathbf{J} = \sum_i m_i \mathbf{r}_i \wedge \dot{\mathbf{r}}_i. \tag{8.12}$$

The rate of change of $\mathbf{J}$ is

$$\dot{\mathbf{J}} = \sum_i m_i \mathbf{r}_i \wedge \ddot{\mathbf{r}}_i = \sum_i \sum_j \mathbf{r}_i \wedge \mathbf{F}_{ij} + \sum_i \mathbf{r}_i \wedge \mathbf{F}_i. \tag{8.13}$$

Now let us examine the contribution to (8.13) from the internal force between a particular pair of particles, say 1 and 2. (See Fig. 8.2.) For simplicity, let us write $\mathbf{F} = \mathbf{F}_{12} = -\mathbf{F}_{21}$ and $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. The contribution consists of two terms, $\mathbf{r}_1 \wedge \mathbf{F}_{12}$ and $\mathbf{r}_2 \wedge \mathbf{F}_{21}$. Thus it is

$$\mathbf{r}_1 \wedge \mathbf{F} - \mathbf{r}_2 \wedge \mathbf{F} = \mathbf{r} \wedge \mathbf{F}. \tag{8.14}$$

This contribution will be zero if $\mathbf{F}$ is a *central* force, parallel to $\mathbf{r}$. Hence if we assume that all the internal forces are central, then the terms in the double sum of (8.13) will cancel in pairs, just as they did in the evaluation of the rate of change of total momentum. The total moment of all the internal forces will be zero. So, when the internal forces are central, the rate of change of angular momentum is equal to the sum of the moments of the external forces,

$$\dot{\mathbf{J}} = \sum_i \mathbf{r}_i \wedge \mathbf{F}_i. \tag{8.15}$$

In particular, for an isolated system, we have the law of *conservation of angular momentum*,

$$\mathbf{J} = \text{constant}. \tag{8.16}$$

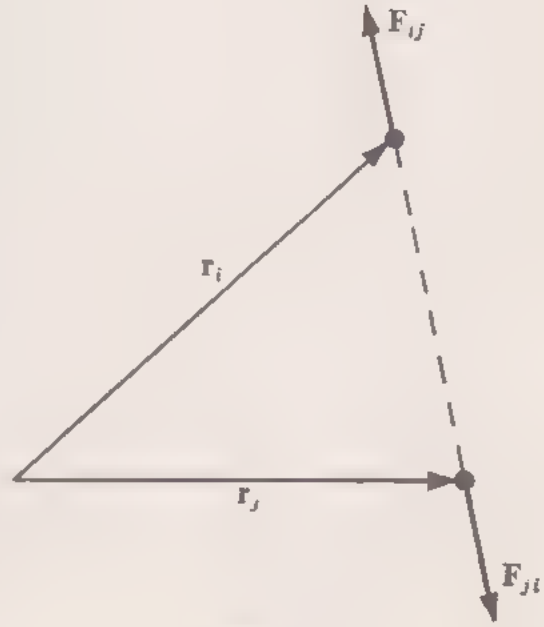


Fig. 8.2

More generally, this is true if all the external forces are directed towards, or away from, the origin.

It is often convenient to separate the contributions to **J** from the centre of mass motion and the relative motion. We define the positions $\mathbf{r}_i^*$ of the particles relative to the centre of mass by

$$\mathbf{r}_i = \mathbf{R}+\mathbf{r}_i^*. \tag{8.17}$$

Clearly, the position of the centre of mass relative to itself is the zero vector, so that

$$\sum_i m_i\mathbf{r}_i^* = \mathbf{0}. \tag{8.18}$$

Now, substituting (8.17) into (8.12), we obtain

$$\mathbf{J} = \Big(\sum_i m_i\Big)\mathbf{R}\wedge\dot{\mathbf{R}}+\Big(\sum_i m_i\mathbf{r}_i^*\Big)\wedge\dot{\mathbf{R}}+\mathbf{R}\wedge\Big(\sum_i m_i\dot{\mathbf{r}}_i^*\Big)+\sum_i m_i\mathbf{r}_i^*\wedge\dot{\mathbf{r}}_i^*.$$

The second and third terms vanish in virtue of (8.18). Thus we can write

$$\mathbf{J} = M\mathbf{R}\wedge\dot{\mathbf{R}}+\mathbf{J}^*, \tag{8.19}$$

where $\mathbf{J}^*$, the angular momentum about the centre of mass, is

$$\mathbf{J}^* = \sum_i m_i\mathbf{r}_i^*\wedge\dot{\mathbf{r}}_i^*. \tag{8.20}$$

It is easy to find the rate of change of $\mathbf{J}^*$. For, by (8.6),

$$\frac{\mathrm{d}}{\mathrm{d}t}(M\mathbf{R}\wedge\dot{\mathbf{R}}) = M\mathbf{R}\wedge\ddot{\mathbf{R}} = \mathbf{R}\wedge\sum_i\mathbf{F}_i. \tag{8.21}$$

Hence, subtracting from (8.15), and using (8.17), we obtain

$$\dot{\mathbf{J}}^* = \sum_i\mathbf{r}_i^*\wedge\mathbf{F}_i.$$

Thus the rate of change of $\mathbf{J}^*$ is equal to the sum of the moments of the external forces about the centre of mass. This is a remarkable result. For, it must be remembered that the centre of mass is not in general moving uniformly. In general, we may take moments about the origin of any inertial frame, but it would be quite wrong to take moments about an accelerated point. Only in the special case where the point is the centre of mass are we allowed to do this.

This result means that in discussing the rotational motion of a body we can ignore the motion of the centre of mass, and treat it as though it were fixed. It is particularly important in the case of rigid bodies, which we discuss in Chapter 10.

For an isolated system, $\mathbf{J}^*$ as well as $\mathbf{J}$ is of course a constant. More generally, $\mathbf{J}^*$ is constant if the external forces have zero total

moment about the centre of mass. For example, for a system of particles in a uniform gravitational field, the resultant force acts at the centre of mass, and $\mathbf{J}^*$ is therefore a constant.

8.4 The Earth–Moon System

As an interesting example of the use of the angular momentum conservation law, we shall consider the system comprising the earth and moon. (We ignore the other planets, and treat the sun as fixed.)

The angular momentum of this system is

$$\mathbf{J} = M\mathbf{R}\wedge\dot{\mathbf{R}}+\mathbf{J}^*, \tag{8.22}$$

where $\mathbf{J}^*$ is the angular momentum about the centre of mass of the system. This angular momentum $\mathbf{J}^*$ can again be separated into an

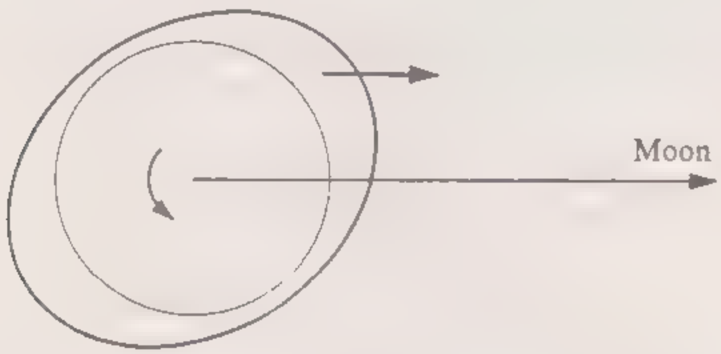


Fig. 8.3

orbital term, due to the motion of the earth and moon around their common centre of mass, and two rotational terms $\mathbf{J}_E{}^*$ and $\mathbf{J}_M{}^*$, due to the rotation of each body around its own centre. The orbital angular momentum, according to (7.15), is $\mu\mathbf{r}\wedge\dot{\mathbf{r}}$. Thus

$$\mathbf{J}^* = \mu\mathbf{r}\wedge\dot{\mathbf{r}}+\mathbf{J}_E^*+\mathbf{J}_M^*. \tag{8.23}$$

Let us first neglect the non-uniformity of the sun's gravitational field over the dimensions of the earth–moon system. Then $\mathbf{J}^*$ as well as $\mathbf{J}$ will be constant. Indeed, to a good approximation, the individual terms of (8.23) are separately constant. However, over long periods of time they do change, for at least two reasons. One is the non-spherical shape of the earth, which as we saw in §6.5 leads to a precession of the moon's orbital plane. This is too small to be of much importance (the non-uniformity of the sun's field produces a much larger effect), but because of angular momentum conservation there is a corresponding precession of the earth's angular momentum $\mathbf{J}_E{}^*$ which is important. We shall discuss this effect further in §10.1. try (6.37)

The second reason is the existence of tidal friction. The dissipation of energy by the tides has the effect of gradually slowing the earth's rotation. We can picture this effect as follows: as the earth rotates, it tries to carry with it the tidal 'bulges', while the moon's attraction is pulling them back into line. (See Fig. 8.3.) Thus there is a couple acting to slow the earth's rotation, and a corresponding equal and opposite couple tending to increase the orbital angular momentum.

To investigate this effect, we shall make the simplest possible approximations. We neglect the rotational angular momentum $\mathbf{J}_M{}^*$ of the moon (which is in fact very small compared to the other terms of (8.23)), and assume that the earth's axis is perpendicular to the orbital plane, so that the remaining terms of (8.23) have the same direction. We also assume that the earth is a uniform sphere of radius r and that the orbit is a circle of radius a. If Ω is the orbital angular velocity, then the orbital angular momentum is $\mu a^2\Omega$; and, if ω is the earth's rotational angular velocity, then (as we show in the next chapter) its angular momentum is $J_E{}^* = \frac{2}{5}mr^2\omega$. Thus

$$J^* = \mu a^2\Omega + \tfrac{2}{5}mr^2\omega. \tag{8.24}$$

Hence we may write the conservation law in the form

$$\frac{5}{2}\frac{\mu}{m}\left(\frac{a}{r}\right)^2\Omega + \omega = \omega_0 = \text{constant.} \tag{8.25}$$

Inserting the values $\mu/m = 0{\cdot}012$, $a/r = 60$, and $\Omega/\omega = 0{\cdot}037$ we find that the first term is at present about four times as large as the second.

Now the orbital angular velocity Ω is related to a by

$$\Omega^2 a^3 = GM. \tag{8.26}$$

Thus as a varies, Ω is proportional to $a^{-3/2}$, and $\omega_0 - \omega$ is proportional to $a^{1/2}$. As ω decreases, a must increase, and therefore Ω also decreases. If we use (8.25) and (8.26) to plot curves of ω and Ω against a, we obtain curves like those shown in Fig. 8.4 (which is not drawn to scale).

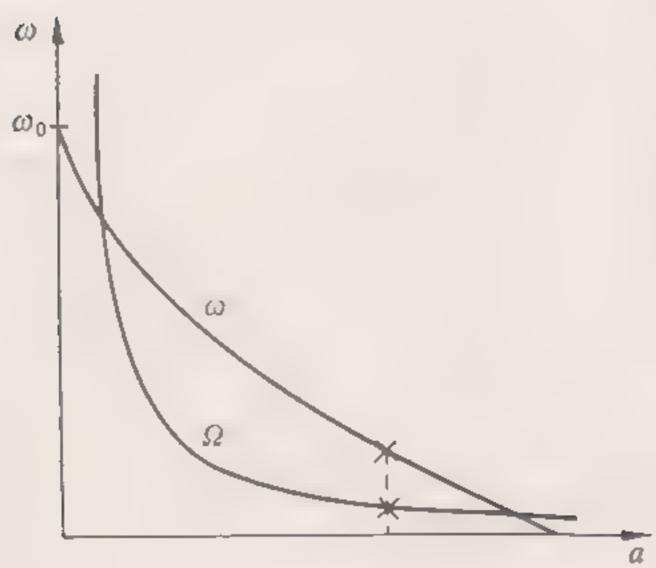


Fig. 8.4

It is important to note that if at any time Ω were actually larger than ω, so that the moon revolved around the earth in less than a day, then the tidal forces would act to increase the angular velocities and decrease the separation. On this simple picture, the smallest distance there can have been between the bodies corresponds to the first point where the curves cross (about 2·5 earth radii, with a common rotation period of under 6 hours). The final state of the system

in the very distant future would correspond to the second point where the curves cross. Then ω and Ω would again be equal, and tidal forces would play no further role. In this final state, the earth would always present the same face to the moon, as the moon already does to the earth. The distance between earth and moon would then be about half as large again as it now is, and the common rotation period (the 'day' or 'month') would be rather more than 50 days.

There are, however, several factors which might modify this picture. Until recently it was generally believed that Mercury and the sun had reached the final state described above, and that Mercury always presented the same face to the sun. In fact, it is now known that the ratio of the orbital and rotation periods is not unity but 3/2. The stability of this configuration depends on the unusually large eccentricity of Mercury's orbit. It is unlikely that such a state could be reached by the earth–moon system.

The 'final' state would in any case not be the end of the evolution of the system, because the *solar* tides would still be active. They would tend to slow the earth's rotation even more, and would therefore gradually pull the moon back towards the earth. In the very remote future the two bodies would collide, although it is probable that the sun will have reached the end of its life before that happens.

Our assumption of uniform density for the earth is clearly wrong. But taking account of the true density distribution would change only the detailed figures, not the general conclusions. However, the density distribution does change from time to time in minor ways (for example, due to the melting of the ice sheets about ten thousand years ago). Such changes may be responsible for some observed short-term fluctuations in the length of the day.

A much more significant simplification in our treatment is the assumption that the moon's orbit is a circle in a plane normal to the earth's axis. In reality, both its eccentricity and its inclination can change with time, and of course the earth's axis also precesses in a complicated way. The effect of tidal energy dissipation is not only to make the moon's orbit expand, but also to make it more eccentric and less inclined. In the distant past, it may have been much more steeply inclined than it now is. Indeed, it has been suggested that the inclination may initially have exceeded 90°, so that the moon was moving in a *retrograde* orbit (from west to east). In such an orbit the tidal forces would act to *reduce* the orbital radius. However, it is rather unlikely that the orbit was in fact retrograde.

The sun has another effect which has been omitted from our discussion. The solar ocean tides are indeed small compared to the lunar ones. However, there are quite substantial semi-diurnal tides in the

earth's atmosphere, noticeable as fluctuations in the barometric pressure at sea level. They are predominantly of thermal, rather than gravitational, origin, due to heating during the hours of daylight, and are very different in character from the ocean tides. The most important component has a period of half a *solar*, rather than lunar, day. It is a little surprising that the dominant period is 12 hours, rather than 24. This may be an example of the phenomenon of resonance: the amplitude of the 12-hour oscillation may be enhanced by being closer to a natural oscillation frequency of the atmosphere than is 24 hours. Whatever the origin of these tides, the sun exerts a gravitational couple on the resulting bulges in the atmosphere. (So does the moon, of course, but its effect averages out to zero over a lunar month.) It happens that the maxima in this case occur *before*, not after, 12 noon and 12 midnight. Thus the couple acts to speed the earth's rotation rather than to slow it. This process has the effect of transferring angular momentum from the earth's orbital motion to its rotation. It therefore tends to decrease the earth's orbital radius, pulling it (immeasurably slowly) closer to the sun. A purely gravitational process could not have this effect, for the energy (kinetic plus potential) is actually increasing. This energy comes from solar heating, with the earth acting as a kind of giant heat engine converting heat to mechanical energy.

It has been suggested that the effect of the sun in speeding the earth's rotation might, over a long period, cancel out the effect of the moon in slowing it, so that the rotation period stayed close to twice a natural resonance period. At the present time the sun's effect is much too small, however, and it seems unlikely that it could have been many times larger in the past. This theory would require a very sharp resonance close to 12 hours, for which evidence is lacking.

8.5 Energy; Conservative Forces

The kinetic energy of our system of particles is

$$T = \sum_i \tfrac{1}{2} m_i \dot{\mathbf{r}}_i^2. \tag{8.27}$$

As in the case of angular momentum, T can be separated into centre-of-mass and relative terms. Substituting (8.17) into (8.27), we again find that the cross terms drop out because of (8.18). Thus we obtain

$$T = \tfrac{1}{2} M\dot{\mathbf{R}}^2 + T^*, \tag{8.28}$$

where T^*, the kinetic energy relative to the centre of mass, is

$$T^* = \sum_i \tfrac{1}{2} m_i \dot{\mathbf{r}}_i^{*2}. \tag{8.29}$$

The relations (8.3), (8.19) and (8.29) show that, exactly as in the two-particle case, the momentum, angular momentum and kinetic energy in any frame are obtained from those in the CM frame by adding the contribution of a particle of mass M located at the centre of mass $\mathbf{R}$.

The rate of change of kinetic energy is

$$\dot{T} = \sum_i m_i \dot{\mathbf{r}}_i \cdot \ddot{\mathbf{r}}_i = \sum_i \sum_j \dot{\mathbf{r}}_i \cdot \mathbf{F}_{ij} + \sum_i \dot{\mathbf{r}}_i \cdot \mathbf{F}_i. \tag{8.30}$$

This equation may be compared with (8.13) for $\dot{\mathbf{J}}$. Let us again examine the contribution from the internal force $\mathbf{F}_{12} = -\mathbf{F}_{21} = \mathbf{F}$ between particles 1 and 2. As in (8.14) it is

$$\dot{\mathbf{r}}_1 \cdot \mathbf{F} - \dot{\mathbf{r}}_2 \cdot \mathbf{F} = \dot{\mathbf{r}} \cdot \mathbf{F}.$$

This is of course the rate at which the force $\mathbf{F}$ does work. However, in contrast to the previous situation, there is in general no reason for this to vanish, and $\dot{T}$ cannot therefore be written in terms of the external forces alone. This is not at all surprising, because in general we must expect there to be some change in the internal potential energy of the system.

We may note that there are certain special cases in which a force does no work, and may be omitted from the energy change equation. This will be the case, for example, if $\dot{\mathbf{r}} = \mathbf{0}$ (e.g. the reaction at a fixed pivot), or if $\mathbf{F}$ is always perpendicular to $\dot{\mathbf{r}}$ (e.g. the magnetic force on a charged particle). An important special case is that of a rigid body, in which the distances between all the particles are fixed. Then r is constant, so that $\mathbf{r} \cdot \dot{\mathbf{r}} = 0$. If the force $\mathbf{F}$ is central, then it has the same direction as $\mathbf{r}$, and is always perpendicular to $\dot{\mathbf{r}}$. Thus a *central* force between particles whose distance apart is fixed does no work.

Now, however, let us make the less restrictive assumption that all the internal forces are conservative. Then the force $\mathbf{F}$ must correspond to a potential energy function of particles 1 and 2, and the rate of working $\dot{\mathbf{r}} \cdot \mathbf{F}$ will be equal to minus the rate of change of this potential energy. Let us denote the total internal potential energy—the sum of these potential energies for all the $\frac{1}{2}N(N-1)$ pairs of particles—by V_{int}. Then clearly the rate of working of all the internal forces will be minus the rate of change of V_{int}. Thus we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(T + V_{\text{int}}) = \sum_i \dot{\mathbf{r}}_i \cdot \mathbf{F}_i. \tag{8.31}$$

The rate of change of the kinetic plus internal potential energy is equal to the rate at which the external forces do work.

We can also find the rate of change of energy relative to the centre of mass. From (8.6) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\tfrac{1}{2}M\dot{\mathbf{R}}^2) = M\dot{\mathbf{R}}\cdot\ddot{\mathbf{R}} = \dot{\mathbf{R}}\cdot\sum_i \mathbf{F}_i.$$

Hence, subtracting from (8.31) and using (8.28) and (8.17), we find

$$\frac{\mathrm{d}}{\mathrm{d}t}(T^*+V_{\text{int}}) = \sum_i \dot{\mathbf{r}}_i^*\cdot\mathbf{F}_i. \tag{8.32}$$

This equation is the analogue of (8.21). Note that V_{int} is a function only of the differences $\mathbf{r}_i-\mathbf{r}_j$, and is therefore unaffected by the choice of reference frame. Thus T^*+V_{int} is the total energy relative to the centre of mass.

Now let us suppose that the external forces are also conservative. Then there must exist a corresponding external potential energy function V_{ext}, whose rate of change is minus the rate of working of the external forces. From (8.31) we then find the law of *conservation of energy*

$$\begin{aligned} T+V &= E = \text{constant}, \\ V &= V_{\text{int}}+V_{\text{ext}}. \end{aligned} \tag{8.33}$$

It is important to notice, however, that V_{ext} depends on the co-ordinates themselves, not merely the co-ordinate differences, and is therefore dependent on the choice of frame, unlike V_{int}. Thus, in general, it is not possible to write down an analogous equation to (8.33) in which only the kinetic energy T^* relative to the centre of mass appears.

from p 164

8.6 Lagrange's Equations

We close this chapter with a brief discussion of the extension of the Lagrangian method of §3.6 to a many-particle system. We consider only the case where all the forces are conservative, so that there exists a total potential energy function V. We saw in §7.1 (see the discussion following (7.10)) that the derivatives of the potential energy function $V_{\text{int}}(\mathbf{r}_1-\mathbf{r}_2)$ corresponding to a two-body force with respect to the co-ordinates of $\mathbf{r}_1$ or $\mathbf{r}_2$ give correctly minus the forces on the corresponding particles. Thus the components of the total force on the ith particle will be minus the derivatives of V with respect to the co-ordinates of $\mathbf{r}_i$. The equations of motion may thus be written

$$m_i\ddot{x}_i = -\frac{\partial V}{\partial x_i}, \tag{8.34}$$

with similar equations for the y and z components.

Now, clearly,

$$\frac{\partial T}{\partial \dot{x}_i} = m_i \dot{x}_i,$$

so that (8.34) are identical with Lagrange's equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i}, \tag{8.35}$$

with $L = T - V$.

Exactly as in §3.6, it follows that the action integral

$$I = \int_{t_0}^{t_1} L \, \mathrm{d}t$$

is stationary under arbitrary variations of the $3N$ co-ordinates r_i which vanish at the limits of integration. Then, again as in §3.6, we may re-express L in terms of any other set of $3N$ co-ordinates $q_1, q_2, \ldots, q_{3N}$. The corresponding equations of motion will be Lagrange's equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) = \frac{\partial L}{\partial q_\alpha}, \qquad \alpha = 1, 2, \ldots, 3N. \tag{8.36}$$

As an example, let us suppose that the external forces are provided by a uniform gravitational field $\mathbf{g}$. The corresponding potential energy function is

$$V_{\text{ext}} = -\sum_i m_i \mathbf{g}\cdot\mathbf{r}_i = -M\mathbf{g}\cdot\mathbf{R}.$$

Now let us choose to write the Lagrangian function in terms of the 3 co-ordinates $\mathbf{R}$ and the co-ordinates $\mathbf{r}_i^*$ (of which only $3N-3$ are independent, because of (8.18)). Then, using (8.28), we find

$$\sum_i m_i \vec{r}_i^{\,*} = \underline{0} \qquad (8\cdot18)$$

$$T = \sum_i \tfrac{1}{2} m_i \dot{\vec{r}}_i^{\,2} \qquad (8\cdot27)$$

$$= \tfrac{1}{2} M \dot{\underline{R}}^2 + T^* \qquad (8\cdot28)$$

$$T^* = \sum_i \tfrac{1}{2} m_i \dot{\underline{r}}_i^{*2} \qquad (8\cdot29)$$

$$L = \tfrac{1}{2}M\dot{\mathbf{R}}^2 + M\mathbf{g}\cdot\mathbf{R} + \sum_i \tfrac{1}{2}m_i\dot{\mathbf{r}}_i^{*2} - V_{\text{int}}. \tag{8.37}$$

Since V_{int} is a function only of the differences $\mathbf{r}_i - \mathbf{r}_j = \mathbf{r}_i^* - \mathbf{r}_j^*$, this Lagrangian function exhibits a complete separation between the terms involving $\mathbf{R}$ and those involving $\mathbf{r}_i^*$, just as in the two-particle case of §7.1. For the case of a uniform gravitational field, the motion of the centre of mass and the relative motion are uncoupled. In particular, there are two separate conservation laws for energy,

$$\tfrac{1}{2}M\dot{\mathbf{R}}^2 - M\mathbf{g}\cdot\mathbf{R} = \text{constant},$$

and

$$T^* + V_{\text{int}} = \text{constant}.$$

As we saw in §8.3, there is also in this case a separate conservation law for angular momentum relative to the centre of mass. (8·16)

It is interesting to note that no observation of the internal motion of a system can ever reveal the presence of a uniform gravitational field, since such a field in no way affects the equations of motion for the relative co-ordinates $\mathbf{r}_i^*$. Note that a laboratory is subjected to other external forces besides the earth's gravitational field, since it is supported by the ground. Indeed, if the uniform field were removed, but the supporting forces retained, there would be no observable difference inside the laboratory (which would of course be accelerated upwards with acceleration g). This was an important consideration in the argument which led to Einstein's general theory of relativity.

8.7 Summary

The centre of mass of any system of particles moves like a particle of mass M acted on by a force equal to the total force on the system. The contributions of this motion to the angular momentum or kinetic energy may be completely separated from the contributions of the relative motion, and $\mathbf{J}$ or T may be written as a sum of two corresponding terms. (Of course, the only contribution to $\mathbf{P}$ comes from the centre-of-mass motion.)

When the internal forces are central, the rate of change of angular momentum is equal to the sum of the moments of the external forces. When they are conservative, the rate of change of the kinetic energy plus internal potential energy is equal to the rate of working of the external forces. In both cases, the same thing is true for the motion relative to the centre of mass.

If the external forces are also central, or conservative, then the total angular momentum, or total energy (including external potential energy), respectively are conserved. In particular, for an isolated system, $\mathbf{P}$, $\mathbf{J}$ and $T+V_{\text{int}}$ are all constants.

PROBLEMS

1 A satellite is orbiting the earth in a circular orbit 230 km above the equator. Calculate the total velocity impulse needed to place it in a synchronous orbit (see Chapter 4, Problem 1), using an intermediate elliptical transfer orbit which just touches both circles. If the final mass to be placed in orbit is 30 kg, and the ejection velocity of the rocket 2·5 km/s, find the necessary initial mass of the rocket.

2 Find the total velocity impulse needed for the trip to Venus described in Chapter 4, Problem 9, including the escape velocity from the earth (a) if the rocket is to pass close by the planet; and (b) if it is to be decelerated to the same velocity, and travel with it.

3 Find the gain in kinetic energy when a rocket emits a small amount of matter. Hence calculate the total energy which must be supplied from chemical or other sources to accelerate the rocket to a given velocity. Show that this is equal to the energy required if an equal amount of matter is ejected while the rocket is held fixed on a test bed.

4 If the residual mass of a rocket, without payload or fuel, is a given fraction λ of the initial mass of rocket plus fuel, show that the total take-off mass required to accelerate a payload m to velocity v is

$$M_0 = m\,\frac{1-\lambda}{e^{-v/u}-\lambda}$$

Find the corresponding expression for a two-stage rocket, in which each stage produces the same velocity impulse. (The first stage rocket is discarded when its fuel is burnt out.) What is the minimum number of stages required if $v = 4u$ and $\lambda = 0{\cdot}15$?

5 A spherical satellite of radius r is moving with velocity v through a uniform tenuous atmosphere of density ρ. Find the retarding force on the satellite if each particle which strikes it (a) adheres to the surface, and (b) bounces off it elastically. Can you explain why the two answers are equal, in terms of the scattering cross-section by a hard sphere?

6 If the orbit of the satellite is highly elliptical, the retarding force is concentrated almost entirely at the lowest point of the orbit (perigee). Replace it by an impulsive force delivered once each orbit, at perigee, and show that the effect is to decrease the orbital period and the apogee distance (greatest distance from the earth's centre), while leaving the perigee distance unaffected. (The orbit therefore becomes more and more circular with time.) What happens to the velocities at perigee and apogee?

7 If the orbit is a circle of radius a, find the rates of change of energy and angular momentum of the satellite. Show that the rates of change of the orbit parameters a and l are equal, so that the orbit remains approximately circular. Show also that the velocity of the satellite must *increase*. Explain the reason for this, using the results of Problem 6.

If the orbit is 500 km above the earth's surface, the mass and radius of the satellite are 20 kg and 50 cm, and $\rho = 10^{-16}$ g cm^{-3}, find the changes in orbital period and height in a year. If the height is 200 km, and $\rho = 10^{-13}$ g cm^{-3}, find the changes in a single orbit.

8 A pair of billiard balls are at rest on a smooth table, and just touching. A third identical ball moving with velocity v perpendicular to their line of centres strikes both simultaneously. The collision is elastic. Find the velocities of all three balls immediately after impact.

9 A uniform spherical cloud of gas of radius a_0 and density ρ_0 is rotating with angular velocity ω_0. It slowly contracts, maintaining uniform density throughout the contraction process, and losing energy by radiation. Show that for energetic reasons there is a minimum radius beyond which the contraction cannot proceed. Evaluate this radius if $\rho_0 = 10^{-22}$ g cm^{-3}, $a_0 = 10^{13}$ km, and the initial angular velocity ω_0 corresponds to a rotation period of 10^{10} years. Find also the rotation period when the contraction reaches this point. (You may assume that for a uniform sphere of mass M and radius a, the angular momentum is $I\omega$, and the kinetic energy is $\frac{1}{2}I\omega^2$, where $I = \frac{2}{5}Ma^2$.)

Chapter 9 Rigid Bodies; Rotation about an Axis

The principal characteristic of a solid body is its rigidity. Under normal circumstances, its size and shape vary only slightly under stress, changes in temperature, and the like. Thus it is natural to consider the idealization of a perfectly rigid body, whose size and shape are permanently fixed. Such a body may be characterized by the requirement that the distance between any two points of the body remains fixed. In this chapter, we shall be concerned with the general principles of the mechanics of rigid bodies, and particularly with the relation between angular momentum and angular velocity for a body rotating about an axis.

9.1 Basic Principles

It will be convenient to simplify the notation of the previous chapter by omitting the particle label i from sums over all particles in the rigid body. Thus, for example, we shall write

$$\mathbf{P} = \sum m\dot{\mathbf{r}}, \qquad \mathbf{J} = \sum m\mathbf{r} \wedge \dot{\mathbf{r}},$$

in place of (8.3) and (8.12).

The motion of the centre of mass of the body is completely specified by (8.6),

$$\dot{\mathbf{P}} = M\ddot{\mathbf{R}} = \sum \mathbf{F}. \tag{9.1}$$

Our main interest in this chapter will be centred on the rotational motion of the body. Let us assume for the moment that the internal forces are central. Then, according to (8.15),

$$\dot{\mathbf{J}} = \sum \mathbf{r} \wedge \mathbf{F}. \tag{9.2}$$

We shall see later that these two equations are sufficient to determine the motion completely.

Under the same assumption of central internal forces, we saw in §8.5 that the internal forces in a rigid body do no work, so that

$$\dot{T} = \sum \dot{\mathbf{r}} \cdot \mathbf{F}. \tag{9.3}$$

This might appear at first sight to be a third independent equation. However, we shall see later that it is actually a consequence of the other two. It is of course particularly useful in the case when the external forces are conservative, since it then leads to the conservation law

$$T + V = E = \text{constant}, \tag{9.4}$$

where V is the external potential energy, previously denoted by V_{ext}. (Note that there is no mention here of internal potential energy: in a rigid body this does not change.)

The assumption that the internal forces are central is much stronger than it need be, and indeed could not be justified from our knowledge of the internal forces in real solids. All we actually require is the validity of equations (9.1) and (9.2), and it is better to regard these as basic assumptions, whose justification lies in the fact that their consequences agree with experiment. The internal forces in real solids cannot adequately be described by classical mechanics, and are certainly not exclusively central—for example, since solids contain moving charges, there are non-central electromagnetic forces. To some extent, the success of the basic assumptions (9.1) and (9.2) may seem rather fortuitous. However, we shall see in Chapter 13 that it is closely related to the relativity principle discussed in Chapter 1.

9.2 Rotation about a Fixed Axis

Let us now apply these basic equations to a rigid body which is free to rotate only about a fixed axis, which for simplicity we take to be the z-axis. We also choose the position of the origin on this axis so that the z co-ordinate of the centre of mass is 0.

In cylindrical polars, the z and ρ co-ordinates of every point are fixed, while the φ co-ordinate varies according to $\dot{\varphi} = \omega$, the angular velocity of the body (not necessarily constant). We examine first the component of angular momentum about the axis of rotation. It is

$$J_z = \sum m\rho v_\varphi = I\omega, \tag{9.5}$$

where

$$I = \sum m\rho^2 \tag{9.6}$$

is the *moment of inertia* about the z-axis. Since I is obviously constant, the z component of (9.2) yields

$$\dot{J}_z = I\dot{\omega} = \sum \rho F_\varphi. \tag{9.7}$$

This equation determines the rate of change of angular velocity, and may be called the equation of motion of the rotating body.

The kinetic energy may also be expressed in terms of I. For

$$T = \sum \tfrac{1}{2}m(\rho\dot{\varphi})^2 = \tfrac{1}{2}I\omega^2. \tag{9.8}$$

The equation (9.3) for the rate of change of kinetic energy is, therefore,

$$\dot{T} = I\omega\dot{\omega} = \sum(\rho\dot{\varphi})F_\varphi = \omega\Sigma\rho F_\varphi.$$

This is clearly a consequence of (9.7), and gives us no additional information.

The momentum equation (9.1) serves to determine the reaction at the axis, which of course has no moment about the axis, and does not appear in (9.7). Let us denote the force on the body at the axis by **Q**, and separate this from the other forces on the body. Then, from (9.1),

$$\dot{\mathbf{P}} = M\ddot{\mathbf{R}} = \mathbf{Q}+\sum\mathbf{F}. \tag{9.9}$$

The centre of mass is of course fixed in the body, so that, by (5.2), $\dot{\mathbf{R}} = \boldsymbol{\omega}\wedge\mathbf{R}$. Differentiating again to find the acceleration of the centre of mass, we obtain

$$\ddot{\mathbf{R}} = \dot{\boldsymbol{\omega}}\wedge\mathbf{R}+\boldsymbol{\omega}\wedge\dot{\mathbf{R}} = \dot{\boldsymbol{\omega}}\wedge\mathbf{R}+\boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{R}).$$

The first term is the tangential acceleration, $R\dot{\omega}$ in the φ direction, and the second the radial acceleration, $-\omega^2R$ in the ρ direction. This equation, together with (9.9), determines **Q**.

As a simple example, let us consider a compound pendulum—a rigid body pivoted about a horizontal axis, and moving under gravity. To be consistent, we shall still take the z-axis to be the axis of rotation, and choose the x-axis vertically downward. (See Fig. 9.1.) Then the force acts at the centre of mass, and has the component $\mathbf{F} = (Mg, 0, 0)$. Thus the equation of motion (9.7) is

$$I\ddot{\varphi} = -MgR\sin\varphi. \tag{9.10}$$

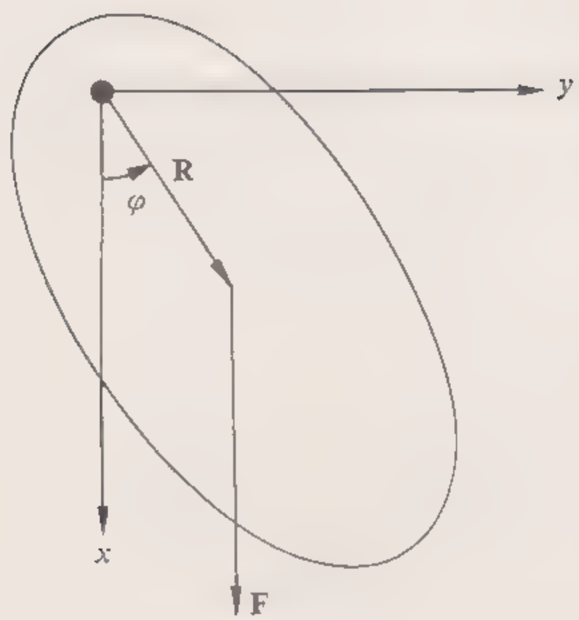


Fig. 9.1

This is identical with the equation of motion of a simple pendulum of length $l = I/MR$. Thus, for example, the period of small oscillations is $2\pi(l/g)^{1/2} = 2\pi(I/MgR)^{1/2}$.

The energy conservation equation is

$$T+V = \tfrac{1}{2}I\dot{\varphi}^2-MgR\cos\varphi = E. \tag{9.11}$$

It may be obtained from (9.10) by multiplying by $\dot{\varphi}$ and integrating. As usual, E is determined by the initial conditions, and (9.11) then serves to fix the angular velocity $\dot{\varphi}$ for any given inclination φ.

The components of the reaction **Q**, obtained from (9.9), are

$$\begin{aligned} Q_z &= 0,\\ Q_\rho &= -Mg\cos\varphi-MR\dot{\varphi}^2,\\ Q_\varphi &= Mg\sin\varphi+MR\ddot{\varphi}. \end{aligned} \tag{9.12}$$

Clearly, $\ddot{\varphi}$ can be expressed in terms of φ using (9.10), and $\dot{\varphi}$ in terms of φ by (9.11), so it is a simple matter to determine **Q** for any value of φ.

9.3 Perpendicular Components of Angular Momentum

We now consider the remaining components of the angular momentum vector **J**. In Cartesian co-ordinates the velocity of a point **r** of the body is given by

$$\dot{x} = -\omega y, \qquad \dot{y} = \omega x, \qquad \dot{z} = 0. \tag{9.13}$$

Thus we find

$$J_x = \sum m(-z\dot{y}) = -\sum mxz\omega,$$

$$J_y = \sum m(z\dot{x}) = -\sum myz\omega.$$

We can write all three components of **J** in the form

$$J_x = I_{xz}\omega, \qquad J_y = I_{yz}\omega, \qquad J_z = I_{zz}\omega, \tag{9.14}$$

where

$$I_{xz} = -\sum mxz, \quad I_{yz} = -\sum myz, \quad I_{zz} = \sum m(x^2+y^2). \tag{9.15}$$

Here I_{zz} is the moment of inertia about the z-axis, previously denoted by I. The quantities I_{xz} and I_{yz} are called *products of inertia*.

At first sight it may seem surprising that **J** has components in directions perpendicular to $\boldsymbol{\omega}$. A simple example may help to clarify the reason for this. Consider a light rigid rod with equal masses m at its two ends, rigidly fixed at an angle θ to an axis through its midpoint. (See Fig. 9.2.) If the positions of the masses are **r** and $-\mathbf{r}$, the total angular momentum is

$$\mathbf{J} = m\mathbf{r}\wedge\dot{\mathbf{r}}+m(-\mathbf{r})\wedge(-\dot{\mathbf{r}}) = 2m\mathbf{r}\wedge(\boldsymbol{\omega}\wedge\mathbf{r}).$$

Clearly, **J** is perpendicular to **r**, as shown in the figure. When the rod is in the xz-plane, the masses are moving in the $\pm y$ directions, and there is a component of angular momentum about the x-axis as well as one about the z-axis.

In this example, the centre of mass lies on the axis. Thus if no external force is applied, the total reaction on the axis is zero. There is, however, a resultant couple on the axis, which is required to balance the couple produced by the centrifugal forces. The magnitude of this couple may be determined from the remaining pair of the equations

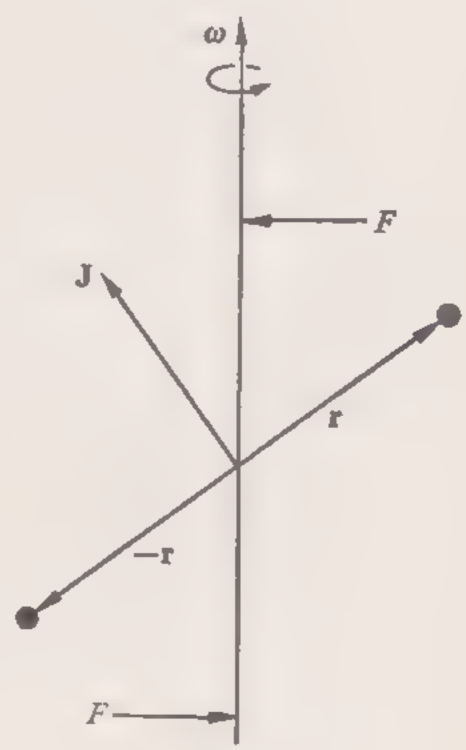


Fig. 9.2

(9.2). From (9.15) we see that, unlike I_{zz}, the products of inertia are not constants, in general. Using (9.13), we find

$$\dot{I}_{xz} = -\omega I_{yz}, \qquad \dot{I}_{yz} = \omega I_{xz}, \qquad \dot{I}_{zz} = 0.$$

Thus, if **G** is the couple on the axis, we have from (9.2)

$$\dot{J}_x = I_{xz}\dot{\omega} - I_{yz}\omega^2 = G_x + \sum(yF_z - zF_y),$$
$$\dot{J}_y = I_{yz}\dot{\omega} + I_{xz}\omega^2 = G_y + \sum(zF_x - xF_z).$$

If there are no external forces, then ω is constant, and **G** precisely balances the centrifugal couple. When the rod is in the xz-plane, the only non-vanishing component is the moment about the y-axis,

$$G_y = I_{xz}\omega^2 = -2mr^2\omega^2 \sin\theta \cos\theta.$$

9.4 Principal Axes of Inertia

We have seen that in general the angular momentum vector **J** is in a different direction from the angular velocity vector $\boldsymbol{\omega}$. There are special cases, however, in which the products of inertia I_{xz} and I_{yz} vanish. Then **J** is also in the z direction. In that case, the z-axis is called a *principal axis of inertia.* When a body is rotating freely about a principal axis through its centre of mass, there is no resultant force or couple on the axis.

The z-axis will be a principal axis, in particular, if the xy-plane is a plane of reflection symmetry; for then the contribution to the products of inertia I_{xz} and I_{yz} from any point (x, y, z) is exactly cancelled by that from the point $(x, y, -z)$. Similarly, it will be a principal axis if it is an axis of rotational symmetry, for then the contribution from (x, y, z) is cancelled by that from $(-x, -y, z)$.

For bodies with three symmetry axes—such as a rectangular parallelepiped or an ellipsoid—there are obviously three perpendicular principal axes, and we shall see later that this is true more generally. In this case, it is clearly an advantage to choose these as our co-ordinate axes. So we shall no longer assume that the axis of rotation is the z-axis but take it to have an arbitrary inclination. Then $\boldsymbol{\omega}$ has three components $(\omega_x, \omega_y, \omega_z)$.

The angular momentum vector **J** is given by

$$\mathbf{J} = \sum m\mathbf{r} \wedge (\boldsymbol{\omega} \wedge \mathbf{r}) = \sum m[r^2\boldsymbol{\omega} - (\mathbf{r}\cdot\boldsymbol{\omega})\mathbf{r}]. \qquad (9.16)$$

Its components are therefore linear functions of the components of $\boldsymbol{\omega}$, which we may write

$$\begin{aligned} J_x &= I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z, \\ J_y &= I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z, \\ J_z &= I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z. \end{aligned} \tag{9.17}$$

The products and moments of inertia are defined as in (9.15).

The nine quantities $I_{xx}, I_{xy}, \ldots, I_{zz}$ may conveniently be exhibited in a square array (or *matrix*)

$$\begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}. \tag{9.18}$$

They may be regarded as components of a single entity $\boldsymbol{\mathsf{I}}$, in much the same way that the quantities J_x, J_y, J_z are regarded as components of the vector $\mathbf{J}$. The entity $\boldsymbol{\mathsf{I}}$ is called a *tensor*, in this case the *inertia tensor*. The elementary properties of tensors are described in Appendix C. However, we shall need to use only one result from this general theory.

Upper case emboldened sans serif 'I'

→ refer to Appendix C.
→ result on page 239

It is obvious from the definition (9.15) that the products of inertia satisfy relations like $I_{xy} = I_{yx}$. The array (9.18) is therefore unchanged by reflection in the leading diagonal. A tensor $\boldsymbol{\mathsf{I}}$ with this property is called *symmetric*.

If the three co-ordinate axes are all axes of symmetry, then all the products of inertia vanish, and $\boldsymbol{\mathsf{I}}$ has the *diagonal* form

$$\begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}.$$

In this case, the relations (9.17) simplify to

$$J_x = I_{xx}\omega_x, \qquad J_y = I_{yy}\omega_y, \qquad J_z = I_{zz}\omega_z. \tag{9.19}$$

Then $\mathbf{J}$ is parallel to $\boldsymbol{\omega}$ if the axis of rotation is any one of the three symmetry axes, but not in general otherwise.

It is shown in Appendix C that for *any* given symmetric tensor one can always find a set of axes with respect to which it is diagonal. Thus for any rigid body we can find three perpendicular axes through any given point which are principal axes of inertia. It will be convenient to introduce three unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ along these axes. Then if we write

The only value here is the use of three orthogonal axes.
The matter of C-16.

$$\boldsymbol{\omega} = \omega_1\mathbf{e}_1 + \omega_2\mathbf{e}_2 + \omega_3\mathbf{e}_3, \tag{9.20}$$

the components of $\mathbf{J}$ in these three directions will be obtained by multiplying the components of $\boldsymbol{\omega}$ by the appropriate moments of inertia, as in (9.19). Thus we obtain

$$\mathbf{J} = I_1\omega_1\mathbf{e}_1 + I_2\omega_2\mathbf{e}_2 + I_3\omega_3\mathbf{e}_3. \tag{9.21}$$

The three diagonal elements of the inertia tensor, I_1, I_2, I_3, are called *principal moments of inertia*. We shall always use a single subscript for the principal moments, to distinguish them from moments of inertia about arbitrary axes.

It is important to realize that the principal axes $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are fixed in the body, not in space, and therefore rotate with it. It is often convenient to use these axes to define our frame of reference, particularly since the principal moments I_1, I_2, I_3 are constants. This is, however, a rotating frame, not an inertial one.

The kinetic energy T may also be expressed in terms of the angular velocity and the inertia tensor. We have

$$T = \sum \tfrac{1}{2}m\dot{\mathbf{r}}^2 = \sum \tfrac{1}{2}m(\boldsymbol{\omega}\wedge\mathbf{r})^2 = \sum \tfrac{1}{2}m[\omega^2 r^2 - (\boldsymbol{\omega}\cdot\mathbf{r})^2],$$

by a standard formula of vector algebra. Comparing with (9.16), we see that

$$T = \tfrac{1}{2}\mathbf{J}\cdot\boldsymbol{\omega}. \tag{9.22}$$

§9.2 Rotation about a fixed axis

(Equations (9.8) and (9.5) provide a special case of this general result.) Thus from (9.20) and (9.21) we find

$$T = \tfrac{1}{2}I_1\omega_1^{\,2} + \tfrac{1}{2}I_2\omega_2^{\,2} + \tfrac{1}{2}I_3\omega_3^{\,2}. \tag{9.23}$$

These equations for rotational motion may be compared with the corresponding ones for translational motion with velocity $\mathbf{v}$, $T = \frac{1}{2}M\mathbf{v}^2 = \frac{1}{2}\mathbf{P}\cdot\mathbf{v}$. The principal difference is that mass, unlike the moments of inertia, has no directional properties, so that the coefficients of $v_x^{\,2}$, $v_y^{\,2}$ and $v_z^{\,2}$ are all equal.

9.5 Shift of Origin

The principal axes and principal moments of inertia refer to a particular choice of origin within the body. When the body is rotating about any axis through this origin, they determine its angular momentum and kinetic energy by (9.21) and (9.23).

When the rigid body is pivoted so that one point is fixed, it is convenient to choose this point to be the origin. If there is no fixed point, we generally choose the origin at the centre of mass. Thus

it is useful to be able to relate the moments and products of inertia about an arbitrary origin to those about the centre of mass.

As usual, we shall distinguish quantities referred to the centre of mass as origin by an asterisk. To find the desired relations, we substitute in (9.15) $\mathbf{r} = \mathbf{R}+\mathbf{r}^*$, and use the relations (8.18) or

$$\sum mx^* = \sum my^* = \sum mz^* = 0.$$

Because of these relations, the cross terms between $\mathbf{R}$ and $\mathbf{r}^*$ drop out, exactly as they did in (8.19) or (8.28). For example,

$$I_{xy} = -\sum m(X+x^*)(Y+y^*) = -MXY-\sum mx^*y^*.$$

The last term is just the product of inertia I^*_{xy} referred to the centre of mass as origin. Thus we obtain relations of the form

$$\begin{aligned} I_{xx} &= M(Y^2+Z^2)+I^*_{xx}, \\ I_{xy} &= -MXY+I^*_{xy}. \end{aligned} \tag{9.24}$$

Note that, as in previous cases, the components of the inertia tensor with respect to an arbitrary origin are obtained from those with respect to the centre of mass by adding the contribution of a particle of mass M at $\mathbf{R}$. (For the moments of inertia, this is known as the *parallel axes theorem*.) Because of this result, it is only necessary, for any given body, to compute the moments and products of inertia with respect to the centre of mass. Those with respect to any other origin are then given by (9.24).

It is important to realize that the principal axes at a given origin are not necessarily parallel to those at the centre of mass. If we choose the axes at the centre of mass to be principal axes, then the products of inertia $I^*_{xy}, \ldots,$ will all be zero, but it is clear from (9.24) that this does not necessarily imply that $I_{xy}, \ldots,$ are zero. In fact, this will be true only if the chosen origin lies on one of the principal axes through the centre of mass, so that two of the three centre-of-mass co-ordinates X, Y, Z, are zero.

9.6 Symmetric Bodies

from p158.

The term *symmetric* applied to a rigid body has a technical significance, and means that two of its principal moments of inertia coincide, say $I_1 = I_2$. This will be true in particular if the third axis $\mathbf{e}_3$ is a symmetry axis of an appropriate type. It may be an axis of cylindrical symmetry—for example, the axis of a spheroid, or a

circular cylinder or cone. More generally, $I_1 = I_2$ if $\mathbf{e}_3$ is an axis of more than two-fold rotational symmetry—for example, the axis of an equilateral triangular prism, or a pyramid with square base. (In the case of the prism, this is not entirely obvious. It follows from the fact that one cannot find a pair of perpendicular axes $\mathbf{e}_1$ and $\mathbf{e}_2$ which are in any way distinguished from other possible pairs.)

For a symmetric body, (9.21) becomes

$$\mathbf{J} = I_1(\omega_1\mathbf{e}_1+\omega_2\mathbf{e}_2)+I_3\omega_3\mathbf{e}_3. \tag{9.25}$$

If $\omega_3 = 0$, then $\mathbf{J} = I_1\boldsymbol{\omega}$, so that $\mathbf{J}$ is parallel to $\boldsymbol{\omega}$. In other words, any axis in the plane of $\mathbf{e}_1$ and $\mathbf{e}_2$ is a principal axis. The symmetry axis $\mathbf{e}_3$ is of course uniquely determined (except for a possible reversal of direction), but the axes $\mathbf{e}_1$ and $\mathbf{e}_2$ are not. We may choose them to be any pair of perpendicular axes in the plane normal to $\mathbf{e}_3$. Indeed, they need not even be fixed in the body, so long as they always remain perpendicular to $\mathbf{e}_3$ and to each other. Equation (9.25) will still hold so long as this condition is satisfied. This freedom of choice will prove to be very useful later.

It can happen that all three principal moments of inertia are equal, $I_1 = I_2 = I_3$. This is the case for a sphere (with origin at the centre), a cube, or a regular tetrahedron (or, indeed, any of the five regular solids). It may of course also happen by an accidental coincidence of the three values. We may call such bodies *totally symmetric.* They have the property that $\mathbf{J} = I_1\boldsymbol{\omega}$, so that every axis is a principal axis, and the choice of $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, is completely arbitrary.

9.7 Calculation of Moments of Inertia

The moments and products of inertia of any body with respect to a given origin may be calculated from the definitions (9.15). For continuous distributions of matter, we must replace the sums by integrals,

$$\begin{aligned} I_{xx} &= \iiint \rho(\mathbf{r})(y^2+z^2)\,\mathrm{d}^3\mathbf{r}, \\ I_{xy} &= \iiint \rho(\mathbf{r})(-xy)\,\mathrm{d}^3\mathbf{r}, \end{aligned} \tag{9.26}$$

etc., where $\rho(\mathbf{r})$ is the density.

If the body has obvious symmetry axes, it is evidently sensible to choose these to be the co-ordinate axes. Then all the products of inertia will vanish, and we need only calculate the three principal

moments. If the principal axes are not obvious, then we must calculate all components of the inertia tensor, and then determine the principal axes and principal moments by the method described in Appendix C. Sometimes the calculation is simplified by dividing up the body into smaller bodies of simpler shape. If their moments of inertia about their own centres of mass are known, we can find the moments about the centre of mass of the whole body using the method of §9.5, and then add the results. C·3.

We shall consider here a class of simple bodies of uniform density ρ and with three perpendicular symmetry planes. The principal axes are then obvious, and we shall take them to be the co-ordinate axes. (Examples of bodies of this type are ellipsoids and rectangular parallelepipeds.) From (9.26) we see that the principal moments of inertia may be written in the form

$$I_1^* = K_y + K_z, \qquad I_2^* = K_z + K_x, \qquad I_3^* = K_x + K_y,$$

where, for example,

$$K_z = \iiint_V \rho z^2 \, dx \, dy \, dz,$$

integrated over the volume V of the body. The mass of the body is of course

$$M = \iiint_V \rho \, dx \, dy \, dz.$$

Now let us denote the lengths of the three symmetry axes by $2a$, $2b$, $2c$, respectively, and consider together all bodies of the same type (e.g. ellipsoids) but with different values of a, b, c. It is easy to find the way in which M and K_z depend on these lengths. Making the substitution $x = a\xi$, $y = b\eta$, $z = c\zeta$, we see that M must be proportional to ρabc, and K_z to ρabc^3. Thus $K_z = \lambda_z Mc^2$, where λ_z is a dimensionless number, the same for all bodies of this type. Hence we obtain *Routh's rule*, which asserts that

$$I_1^* = M(\lambda_y b^2 + \lambda_z c^2), \tag{9.27}$$

with similar expressions for the other two principal moments. The values of the constants λ may be found by examining the special case $a = b = c = 1$. For example, for a sphere of unit radius,

$$K_z = \rho \iiint z^2 \, dx \, dy \, dz = \rho \int_{-1}^{1} z^2 \pi (1 - z^2) \, dz = \frac{4\pi}{15} \rho.$$

Since also $M = 4\pi\rho/3$, we have $\lambda_z = \frac{1}{5}$. Thus for an *ellipsoid*

$$\lambda_x = \lambda_y = \lambda_z = \tfrac{1}{5}.$$

A similar calculation shows that for a *rectangular parallelepiped*,

$$\lambda_x = \lambda_y = \lambda_z = \tfrac{1}{3}.$$

As an example of a body for which the three constants are not all equal, we may quote the *elliptic cylinder* of length $2c$, for which

$$\lambda_x = \lambda_y = \tfrac{1}{4}, \qquad \lambda_z = \tfrac{1}{3}.$$

All three examples may be summarized by saying that λ is $\frac{1}{3}$ for 'rectangular' axes, $\frac{1}{4}$ for 'elliptic' axes, and $\frac{1}{5}$ for 'ellipsoidal' axes.

This formula covers several special cases of particular interest. A sphere is of course an ellipsoid with $a = b = c$, and each of its three principal moments of inertia is $\frac{2}{5}Ma^2$. Similarly, a cube is a parallelepiped with $a = b = c$, and each of its principal moments is $\frac{2}{3}Ma^2$. A parallelepiped with $c = 0$ is a flat rectangular plate. Its principal moments are

$$I_1^* = \tfrac{1}{3}Mb^2, \qquad I_2^* = \tfrac{1}{3}Ma^2, \qquad I_3^* = \tfrac{1}{3}M(a^2+b^2).$$

Similarly, a flat circular plate is a cylinder with $a = b$, and $c = 0$. In that case,

$$I_1^* = I_2^* = \tfrac{1}{4}Ma^2, \qquad I_3^* = \tfrac{1}{2}Ma^2.$$

Finally, a thin rod is a limiting case of either a parallelepiped or cylinder with $a = b = 0$. Its principal moments are

$$I_1^* = I_2^* = \tfrac{1}{3}Mc^2, \qquad I_3^* = 0.$$

9.8 Summary

The basic equations determining the motion of a rigid body are the momentum and angular momentum equations (9.1) and (9.2). When the motion is confined to rotation about a fixed axis, the equation of motion is the component of the angular momentum equation about the axis; the other equations determine the resultant force and couple on the axis. In general, the motion of the centre of mass is determined by the momentum equation, and the rotational motion by the angular momentum equation.

The relation between angular momentum and angular velocity is described by the inertia tensor. It is always possible to find a set of principal axes (fixed in the body) for which all the products of inertia vanish. Then **J** is obtained from $\boldsymbol{\omega}$ simply by multiplying each component by the appropriate principal moment. Similarly, the kinetic

energy T is the sum of the three terms corresponding to rotation about each of the principal axes.

PROBLEMS

1 A uniform cube of edge $2a$ is suspended from a horizontal axis along one edge. Find the length of the equivalent simple pendulum. If the cube is released from rest with its centre of mass level with the axis, find its angular velocity when it reaches the lowest point. Find also the horizontal and vertical components of the reaction on the axis as a function of the inclination. How do these compare with the corresponding values for the equivalent simple pendulum, and why are they different?

2 A flat rectangular plate of dimensions 6 cm $\times$ 8 cm, and mass 20 g, is pivoted about an axis in its plane, and lying along one diagonal. It is spinning about the axis at 120 r.p.m. The axis is of length 12 cm, and is held vertical by bearings at its ends. Find the horizontal component of the force on each bearing.

3 An insect of mass 100 mg is sitting on the edge of a flat uniform disc of mass 3 g, and radius 5 cm, which is rotating at 65 r.p.m. The insect crawls in towards the centre of the disc. Find the angular velocity when it reaches it, and the gain in kinetic energy. Where does this kinetic energy come from, and what happens to it when the insect crawls back out to the edge?

4 A uniform cube of edge $2a$ is sliding with velocity v on a smooth horizontal table, when its leading edge is suddenly brought to rest by a small ridge on the table. Find the angular velocity immediately after impact, and the fractional loss of kinetic energy. Determine the minimum value of v for which the cube topples over rather than falling back.

5 A pendulum consists of a light rigid rod of length 25 cm, with two identical uniform spheres of radius 5 cm attached one on either side of its lower end, so that in equilibrium their centres are level with it. Find the period of small oscillations (a) in the direction of the line of centres, and (b) in the perpendicular direction.

6 The star of Chapter 4, Problem 6, is rotating with a period of one year. Assuming that the trapped particles affect the total mass, but not its distribution, estimate the time required for the rotation period to increase by one day.

7 Calculate the principal moments of inertia of a cone of vertical height h, and base radius a, about its vertex. For what value of the ratio h/a is every axis through the vertex a principal axis? For this case, find the position of the centre of mass, and the principal moments of inertia about it.

8 A spaceship weighing 3 tons has the form of a hollow sphere, with inner radius 2·5 m, and outer radius 3 m. Its orientation in space is controlled by a uniform circular flywheel of mass 10 kg and radius 10 cm. If the flywheel is set spinning at 2000 r.p.m., find how long it takes for the spaceship to rotate through 1°. Find also the energy dissipated in this manoeuvre.

9 A thin hollow cylinder of radius a is balanced on a horizontal knife edge, with its axis parallel to it. It is given a small displacement. Calculate the angular displacement at the moment when the cylinder ceases to touch the knife edge (when the radial component of the reaction falls to zero). Show that it does not touch the knife edge again in the subsequent free fall.

Chapter 10 Dynamics of a Rigid Body

In the last chapter, we discussed the general principles governing the motion of a rigid body, and the special case of rotation about a fixed axis. We shall now go on to discuss the general motion of a rigid body.

10.1 Effect of a Small Force on the Axis

We begin by considering a simple, but very important, type of motion. We suppose that the rigid body is free to rotate about a fixed smooth pivot, and that initially it is rotating freely about a principal axis, say $\mathbf{e}_3$. Then, if the angular velocity is $\boldsymbol{\omega} = \omega\mathbf{e}_3$, the angular momentum will be

$$\mathbf{J} = I_3\boldsymbol{\omega}.$$

So long as no external force acts on the body, the angular momentum equation (9.2) tells us that

$$\dot{\mathbf{J}} = I_3\dot{\boldsymbol{\omega}} = \mathbf{0},$$

so that the axis will remain fixed in space, and the angular velocity will be constant. (This would not be true if the axis of rotation were not a principal axis. We shall see what happens in this case later.)

Now suppose that a small force $\mathbf{F}$ is applied to the axis at a point $\mathbf{r}$. (See Fig. 10.1.) Then the equation of motion becomes

$$\dot{\mathbf{J}} = \mathbf{r} \wedge \mathbf{F}. \tag{10.1}$$

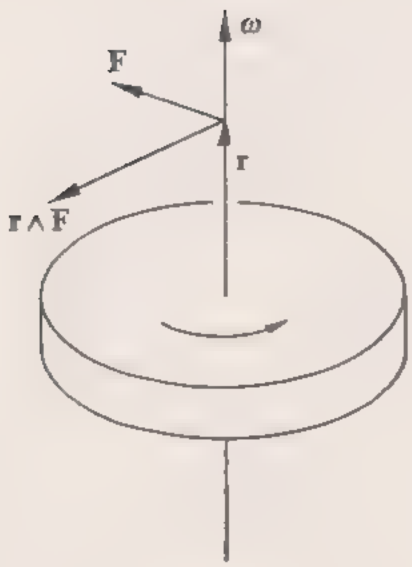


Fig. 10.1

This force will cause the axis to change direction, and the body will therefore acquire a small component of angular velocity perpendicular to its axis $\mathbf{e}_3$. However, provided the force is small enough, the angular velocity with which the axis moves will be small in comparison to the angular velocity of rotation about the axis. In that case, we may neglect the angular momentum components normal to the axis, and again write

$$\dot{\mathbf{J}} = I_3\dot{\boldsymbol{\omega}} = \mathbf{r} \wedge \mathbf{F}. \tag{10.2}$$

Since the moment $\mathbf{r} \wedge \mathbf{F}$ is perpendicular to $\boldsymbol{\omega}$, the magnitude of the angular velocity remains unchanged. (Recall that $\mathrm{d}(\omega^2)/\mathrm{d}t = 2\boldsymbol{\omega}\cdot\dot{\boldsymbol{\omega}}$.) However, its direction does change, and the axis will move in the direction of $\mathbf{r} \wedge \mathbf{F}$, that is, *perpendicular* to the applied force.

This effect may be rather surprising at first sight. However, it is really quite familiar to anyone who has ridden a bicycle. The effect

of leaning over to one side is not to make the rider fall off—as it would be if the wheels were not rotating—but to make the bicycle turn a corner. The force in this case is the force of gravity, which produces a moment about a horizontal axis, as shown in Fig. 10.2. Thus, in a small time interval dt, the angular velocity $\boldsymbol{\omega}$ of a wheel acquires a small additional horizontal component $d\boldsymbol{\omega}$. This shows that the axis of the wheel must change direction. (Note that turning a corner on skis, where no rotating bodies are involved, requires a quite different technique.)

If the force $\mathbf{F}$ is constant in magnitude and direction, it leads to a precession of the axis of rotation, very similar to the Larmor precession we discussed in §5.6, or the precession of a satellite orbit discussed in §6.5. Remembering that both $\mathbf{r}$ and $\boldsymbol{\omega}$ have the common direction $\mathbf{e}_3$, we can rewrite (10.2) in the form

The shape of the Earth (Satellite Orbit) p101

$$I_3\omega\dot{\mathbf{e}}_3 = -r\mathbf{F}\wedge\mathbf{e}_3.$$

This equation is of a form we have encountered several times already. It may be written

$$\dot{\mathbf{e}}_3 = \boldsymbol{\Omega}\wedge\mathbf{e}_3, \tag{10.3}$$

where

$$\boldsymbol{\Omega} = -\frac{r\mathbf{F}}{I_3\omega}. \tag{10.4}$$

Thus (see §5.1)*, it describes a vector $\mathbf{e}_3$ rotating with the constant angular velocity $\boldsymbol{\Omega}$. The axis of the body therefore precesses around the direction of $\mathbf{F}$.

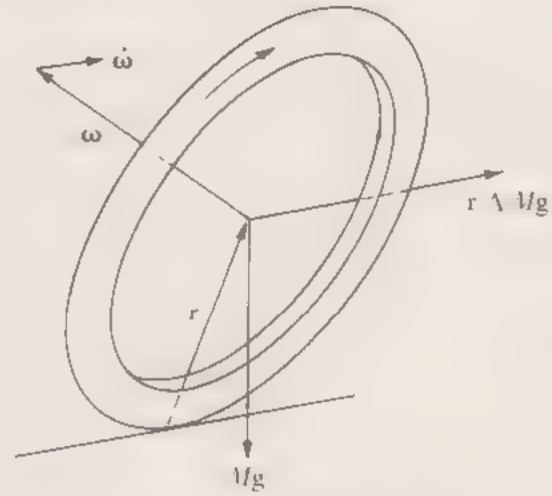


Fig. 10.2

* Angular Velocity; Rate of Change of a Vector
Is there a rotating frame involved?

A familiar example of this effect is the spinning top (which we discuss in more detail in Chapters 11 and 13). The force in this case is the force of gravity $\mathbf{F} = -Mg\mathbf{k}$. A top spinning with angular velocity ω about its axis will therefore precess around the vertical with precessional angular velocity

$$\Omega = \frac{Mgr}{I_3\omega}. \tag{10.5}$$

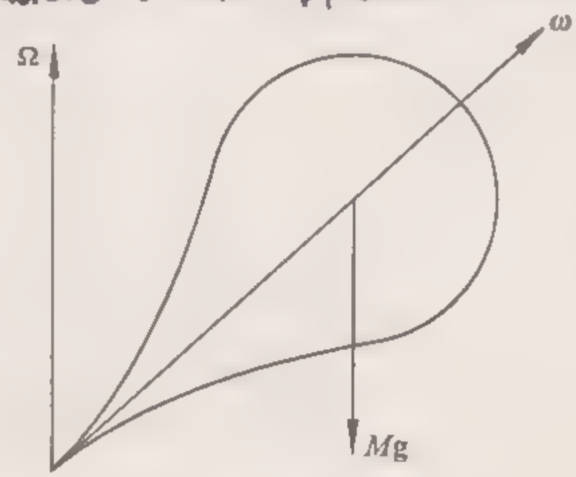


Fig. 10.3

(See Fig. 10.3.) Note that Ω is independent of the inclination of the axis to the vertical.

Our treatment here is valid only if $\Omega \ll \omega$, or, equivalently, if $Mgr \ll I_3\omega^2$. In other words, we require the rotational kinetic energy to be much greater than the possible changes in gravitational potential energy. We shall discuss what happens when this condition is not satisfied later.

Note that the precessional angular velocity (10.4) is inversely

proportional both to the mass of the body and to its rotational angular velocity ω. Thus, to minimize the effect of a given force we should use a heavy, rapidly spinning body.

The great stability of rapidly rotating bodies is the basis of the *gyroscope*. Essentially, this consists of a spinning body suspended in such a way that its axis is free to rotate relative to its support. The bearings are made as nearly frictionless as possible, to minimize the torques on the gyroscope. Then, no matter how we turn the support, the axis of the gyroscope will remain pointing very closely to the same direction in space.

The gyroscope is particularly useful for navigational instruments in aircraft. It may be employed, for example, to provide an 'artificial horizon' which allows the pilot to fly a level course even in cloud (where our normal sense of balance is notoriously unreliable). A similar instrument, in which however the axis of the gyroscope is free to rotate only in a plane, serves as a direction indicator.

Precession of the Equinoxes. The nonspherical shape of the earth leads to another example of this phenomenon of precession. This effect is closely related to the precession of a satellite orbit discussed in §6.5. Since the earth exerts a moment on the satellite, the satellite exerts an equal and opposite moment on the earth. For a small satellite, this is of course quite negligible, but a similar moment is exerted by both the sun and the moon, and although the effect is still quite small it leads to observable changes over long periods of time.

Let us consider for example the effect of the sun. Because it attracts the nearer equatorial bulge more strongly than the farther one, it exerts a couple on the earth's axis, in a direction which would tend (if the earth were not rotating) to align it with the normal to the plane of the earth's orbit (called the *ecliptic* plane, because it is the plane in which eclipses occur). (See Fig. 10.4.) However, because of the earth's rotation, the actual effect is to make the axis precess around the normal to the ecliptic plane. Since the equator is inclined to the ecliptic at about 23·5°, the axis therefore describes a cone in space with semi-vertical angle 23·5°.

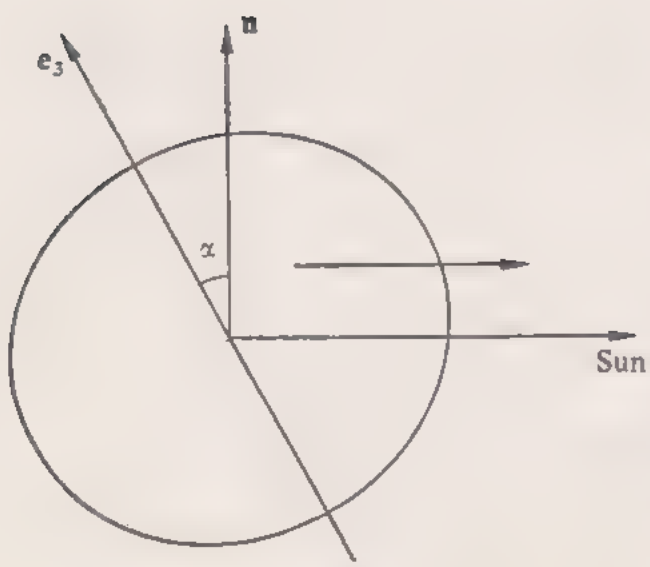


Fig. 10.4

It turns out that this effect depends on the same parameter * ((6.40) or (6.41)) which appeared in the discussion of the tides. Thus, as in that case, the effect of the moon is rather more than twice as large as that of the sun (though somewhat complicated by the fact that the moon's orbital plane is not fixed in space). The two bodies together lead to a precessional angular velocity of about 50″ per year, which corresponds to a precessional period of 26 000 years. The effect means that the pole of the earth's axis moves relative to the fixed stars, so that 13 000 years ago our present pole star was some

* $\frac{mr^3}{Ma^3}$

47° away from the pole. (The irregularity of the moon's motion, which is due to the moment exerted by the sun on the earth–moon system, leads to a superimposed wobble of the axis with an amplitude of 9″ and a period of 18·6 years.)

10.2 Instantaneous Angular Velocity

We shall now turn to the problem of the general motion of a rigid body, without assuming that it is spinning about a principal axis, or that the forces acting are small. In this section we shall discuss the concept of angular velocity, and show that, no matter how the body is moving, it is always possible to define an *instantaneous* angular velocity vector $\boldsymbol{\omega}$ (which in general is constant in neither direction nor magnitude).

We consider first a rigid body free to rotate about a fixed pivot. Then the position of every point of the body is fixed if we specify the directions of the three principal axes $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. (This would be true for any three axes fixed in the body, but it will be convenient later if we choose them to be the principal axes.)

Since the position of any point is fixed relative to these axes, its velocity is determined by the three velocities $\dot{\mathbf{e}}_1$, $\dot{\mathbf{e}}_2$, $\dot{\mathbf{e}}_3$. In fact, if the point in question is

$$\mathbf{r} = r_1\mathbf{e}_1 + r_2\mathbf{e}_2 + r_3\mathbf{e}_3, \tag{10.6}$$

$$= (\underline{r}\cdot\underline{e}_1)\underline{e}_1 + (\underline{r}\cdot\underline{e}_2)\underline{e}_2 + (\underline{r}\cdot\underline{e}_3)\underline{e}_3$$

$$= \underline{r}\cdot(\underline{e}_1\underline{e}_1 + \underline{e}_2\underline{e}_2 + \underline{e}_3\underline{e}_3)$$

$$= \underline{r}\cdot\underline{\underline{1}} = \underline{\underline{R}} \qquad (C\cdot16)$$

then r_1, r_2, r_3 are constants, and

$$\dot{\mathbf{r}} = r_1\dot{\mathbf{e}}_1 + r_2\dot{\mathbf{e}}_2 + r_3\dot{\mathbf{e}}_3. \tag{10.7}$$

$$= (\underline{r}\cdot\underline{e}_1)\dot{\underline{e}}_1 + (\underline{r}\cdot\underline{e}_2)\dot{\underline{e}}_2 + (\underline{r}\cdot\underline{e}_3)\dot{\underline{e}}_3$$

$$=$$

We now wish to show that, no matter how the body is moving, we can always define an *instantaneous* angular velocity $\boldsymbol{\omega}$. Since $\mathbf{e}_1$ is a unit vector, $\mathbf{e}_1^2 = 1$, and we obtain by differentiation

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{e}_1^2) = 2\mathbf{e}_1\cdot\dot{\mathbf{e}}_1 = 0.$$

It follows that $\dot{\mathbf{e}}_1$ is perpendicular to $\mathbf{e}_1$, and has the form

$$\dot{\mathbf{e}}_1 = a_{12}\mathbf{e}_2 + a_{13}\mathbf{e}_3. \tag{10.8}$$

Moreover, since $\mathbf{e}_1$ and $\mathbf{e}_2$ are always perpendicular,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{e}_1\cdot\mathbf{e}_2) = \dot{\mathbf{e}}_1\cdot\mathbf{e}_2 + \mathbf{e}_1\cdot\dot{\mathbf{e}}_2 = 0,$$

$$\begin{bmatrix} \omega_1 & 0 & 0 \\ 0 & \omega_2 & 0 \\ 0 & 0 & \omega_3 \end{bmatrix} \Longleftrightarrow \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & \omega_1 \\ -\omega_2 & -\omega_1 & 0 \end{bmatrix}$$

whence $a_{12}+a_{21} = 0$. Thus we can define a vector $\boldsymbol{\omega}$ by

$$\begin{aligned} \omega_1 &= -a_{23} = a_{32}, \\ \omega_2 &= -a_{31} = a_{13}, \\ \omega_3 &= -a_{12} = a_{21}. \end{aligned} \tag{10.9}$$

With this definition, (10.8) becomes simply

$$\dot{\mathbf{e}}_1 = \boldsymbol{\omega} \wedge \mathbf{e}_1. \tag{10.10}$$

Substituting this, and the two similar equations, in (10.7), we find that the instantaneous velocity of any point in the body is given by

$$\dot{\mathbf{r}} = \boldsymbol{\omega} \wedge \mathbf{r}. \tag{10.11}$$

Thus we have shown that an instantaneous angular velocity vector $\boldsymbol{\omega}$ always exists. In general, it is of course not a constant. We note that the direction of $\boldsymbol{\omega}$ is the instantaneous axis of rotation: for points $\mathbf{r}$ lying on this axis, the instantaneous velocity is zero.

Now let us consider the general case, in which no point of the body is permanently fixed. Then we may specify the position of the body by the position of its centre of mass (or of any other designated point), and by the orientation of the principal axes at that point. Clearly, we can define as before an angular velocity vector $\boldsymbol{\omega}$ relative to the centre of mass. If $\dot{\mathbf{R}}$ is the velocity of the centre of mass, then the instantaneous velocity of any point whose position relative to the centre of mass is $\mathbf{r}^*$ is

$$\dot{\mathbf{r}} = \dot{\mathbf{R}} + \dot{\mathbf{r}}^* = \dot{\mathbf{R}} + \boldsymbol{\omega} \wedge \mathbf{r}^*. \tag{10.12}$$

We have not attached an asterisk to $\boldsymbol{\omega}$, because the angular velocity is in fact independent of the choice of origin. For example, if the body is rotating with angular velocity $\boldsymbol{\omega}$ about a fixed pivot, then

$$\dot{\mathbf{r}} = \boldsymbol{\omega} \wedge \mathbf{r} = \boldsymbol{\omega} \wedge \mathbf{R} + \boldsymbol{\omega} \wedge \mathbf{r}^*. \tag{10.13}$$

But the instantaneous velocity of the centre of mass is given by $\dot{\mathbf{R}} = \boldsymbol{\omega} \wedge \mathbf{R}$, whence, substracting from (10.13),

$$\dot{\mathbf{r}}^* = \boldsymbol{\omega} \wedge \mathbf{r}^*.$$

This shows that $\boldsymbol{\omega}$ is also the instantaneous angular velocity of the motion relative to the centre of mass.

The equation of motion of a rigid body rotating about a fixed pivot is the angular momentum equation (9.2),

$$\dot{\mathbf{J}} = \sum \mathbf{r} \wedge \mathbf{F}, \tag{10.14}$$

where $\mathbf{J}$ is related to $\boldsymbol{\omega}$ by

$$\mathbf{J} = I_1\omega_1\mathbf{e}_1 + I_2\omega_2\mathbf{e}_2 + I_3\omega_3\mathbf{e}_3.$$

(10·14b) as (9·21)

As in the previous chapter, the momentum equation (9.1) serves to determine the reaction at the pivot.

In the general case, we have to determine both the velocity of the centre of mass and the angular velocity. The motion of the centre of mass is described by the momentum equation

$$\dot{\mathbf{P}} = M\ddot{\mathbf{R}} = \sum \mathbf{F}, \tag{10.15}$$

and the rotational motion by

$$\dot{\mathbf{J}}^* = \sum \mathbf{r}^* \wedge \mathbf{F}. \tag{10.16}$$

The angular momentum about the centre of mass is related to the angular velocity by

$$\mathbf{J}^* = I_1^*\omega_1\mathbf{e}_1^* + I_2^*\omega_2\mathbf{e}_2^* + I_3^*\omega_3\mathbf{e}_3^*.$$

10.3 Stability of Rotation about a Principal Axis

The principal axes $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ rotate with the rigid body. Thus, if we wish to use directly the expression for $\mathbf{J}$ in terms of its components with respect to these axes, we have to remember that they constitute a rotating frame. In this section, we shall return to the notation of Chapter 5, and distinguish the absolute rate of change $\mathrm{d}\mathbf{J}/\mathrm{d}t$ from the relative rate of change $\dot{\mathbf{J}}$. The equation of motion (10.14) refers of course to the absolute rate of change, so we must write it as

$$\frac{\mathrm{d}\mathbf{J}}{\mathrm{d}t} = \sum \mathbf{r} \wedge \mathbf{F} = \mathbf{G}, \tag{10.17}$$

say. (The argument here may be applied equally to a rigid body rotating about a fixed pivot, or to the rotation about the centre of mass.)

The relative rate of change is

$$\dot{\mathbf{J}} = I_1\dot{\omega}_1\mathbf{e}_1 + I_2\dot{\omega}_2\mathbf{e}_2 + I_3\dot{\omega}_3\mathbf{e}_3,$$

since the principal moments of inertia are constants. The two rates of change are related by

$$\frac{\mathrm{d}\mathbf{J}}{\mathrm{d}t} = \dot{\mathbf{J}} + \boldsymbol{\omega} \wedge \mathbf{J}.$$

Substituting in (10.17), we obtain

$$\dot{\mathbf{J}} + \boldsymbol{\omega} \wedge \mathbf{J} = \mathbf{G},$$

or, in terms of components,

$$I_3\dot{\omega}_3+(I_2-I_1)\omega_1\omega_2 = G_3, \tag{10.18}$$

and two similar equations obtained by cyclic permutation of 1, 2, 3.

The equation (10.18) may be solved, in principle, to give the angular velocity components as functions of time. However, when there are external forces, it is not a particularly useful equation, because these forces are usually specified in terms of their components with respect to a fixed set of axes. Even if the external force **F** is a constant, its components F_1, F_2, F_3 are variable, and depend on the unknown orientation of the body. For this kind of problem, an alternative method of solution, using Lagrange's equations (to be described in the next chapter), is usually preferable.

We shall confine our discussion here to the case where there are no external forces, so that the right side of (10.18) vanishes. If the body is initially rotating about the principal axis $\mathbf{e}_3$, so that $\omega_1 = \omega_2 = 0$, then we see from (10.18) that ω_3 is a constant. The other two equations show similarly that ω_1 and ω_2 remain zero. Thus we have verified our earlier assertion that a body rotating freely about a principal axis will continue to rotate with constant angular velocity.

Now, however, we wish to investigate the stability of this motion; that is, we ask what happens if the body is given a small displacement, so that its axis of rotation no longer precisely coincides with $\mathbf{e}_3$. Under these circumstances, ω_1 and ω_2 will be small, but not precisely zero. If the displacement is small enough, we may neglect the product $\omega_1\omega_2$, so that from (10.18) we again learn that ω_3 is a constant. The other two equations are

$$I_1\dot{\omega}_1+(I_3-I_2)\omega_2\omega_3 = 0,$$

$$I_2\dot{\omega}_2+(I_1-I_3)\omega_3\omega_1 = 0.$$

We can solve these equations by looking for solutions of the form

$$\omega_1 = a_1 e^{pt}, \qquad \omega_2 = a_2 e^{pt},$$

where a_1 and a_2 are constants. (We shall discuss the general problem of stability by a similar method in the next chapter.) Substituting, and eliminating the ratio a_1/a_2, we obtain

$$p^2 = \frac{(I_3-I_2)(I_1-I_3)}{I_1I_2}\,\omega_3{}^2.$$

The denominator is obviously positive, as is $\omega_3{}^2$. Thus, if $I_3 > I_1$ and $I_3 > I_2$, or if $I_3 < I_1$ and $I_3 < I_2$, the two roots for p are pure

imaginary, and we have an oscillatory solution. On the other hand, if $I_1 > I_3 > I_2$ or $I_1 < I_3 < I_2$, then the roots are real, and the values of ω_1 and ω_2 will in general increase exponentially with time.

Therefore, we may conclude that the rotation about the axis $\mathbf{e}_3$ is stable if I_3 is either the largest or the smallest of the three principal moments, but not if it is the middle one. This interesting result is quite easy to verify. For example, if one throws a match-box in the air, it is not hard to get it to spin about its shortest or longest axis, but it will not spin stably about the other one.

10.4 Euler's Angles

The orientation of a rigid body about a fixed point—or about its centre of mass—must be specified by three angles. These may be chosen in various ways, but one convenient choice is the set of angles known as Euler's angles. Two of these angles are required to specify the direction of one of the axes, say $\mathbf{e}_3$—they are just the polar angles θ, φ—and the third specifies the angle through which the body has been rotated from a standard position about this axis.

We can reach an arbitrary orientation by starting with the body in a standard position, and making three successive rotations. We suppose that initially the axes $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ of the body coincide with the fixed axes $\mathbf{i}, \mathbf{j}, \mathbf{k}$. The first two rotations are designed to bring the axis $\mathbf{e}_3$ to its required position, specified by the polar angles θ, φ. First, we make a rotation through an angle φ about the axis $\mathbf{k}$. (See Fig. 10.5.) This brings the three axes into the positions $\mathbf{e}_1''$, $\mathbf{e}_2'$, $\mathbf{k}$. Next, we make a rotation through an angle θ about the axis $\mathbf{e}_2'$, bringing the axes to the positions $\mathbf{e}_1'$, $\mathbf{e}_2'$, $\mathbf{e}_3$. Finally, we make a rotation through an angle ψ about the axis $\mathbf{e}_3$. This brings all three axes to their required positions $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

Since the three Euler angles φ, θ, ψ fix the orientations of the three axes $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, they specify completely the orientation of the rigid body.

The angular velocity of the rigid body is clearly determined by the rates of change of these three angles. A small change in φ corresponds to a rotation about $\mathbf{k}$. Hence, if φ is changing at a rate $\dot{\varphi}$, and the other angles are constant, the angular velocity is $\dot{\varphi}\mathbf{k}$. Similarly, if θ or ψ is changing, the angular velocity is $\dot{\theta}\mathbf{e}_2'$ or $\dot{\psi}\mathbf{e}_3$. When all three angles are changing, the angular velocity is the sum of these three contributions,

$$\boldsymbol{\omega} = \dot{\varphi}\mathbf{k} + \dot{\theta}\mathbf{e}_2' + \dot{\psi}\mathbf{e}_3. \tag{10.19}$$

(9·21) is repeated at (10·146)
(9·23) refered to.

In order to find the angular momentum or kinetic energy, using (9.21) or (9.23), we should have to find the components of $\boldsymbol{\omega}$ in the directions of the three principal axes. This is not difficult, but we shall not go through the derivation in the general case. Instead, we limit the discussion to the case of a *symmetric body*, with symmetry axis $\mathbf{e}_3$, and $I_1 = I_2$. Then, according to the discussion of §9.6, any pair of axes in the plane of $\mathbf{e}_1$ and $\mathbf{e}_2$ will serve as principal axes, together with $\mathbf{e}_3$. It is not necessary that the axes we choose should be fixed in the body, so long as they are always principal axes. In particular, we may use the axes $\mathbf{e}_1'$, $\mathbf{e}_2'$. This choice has the

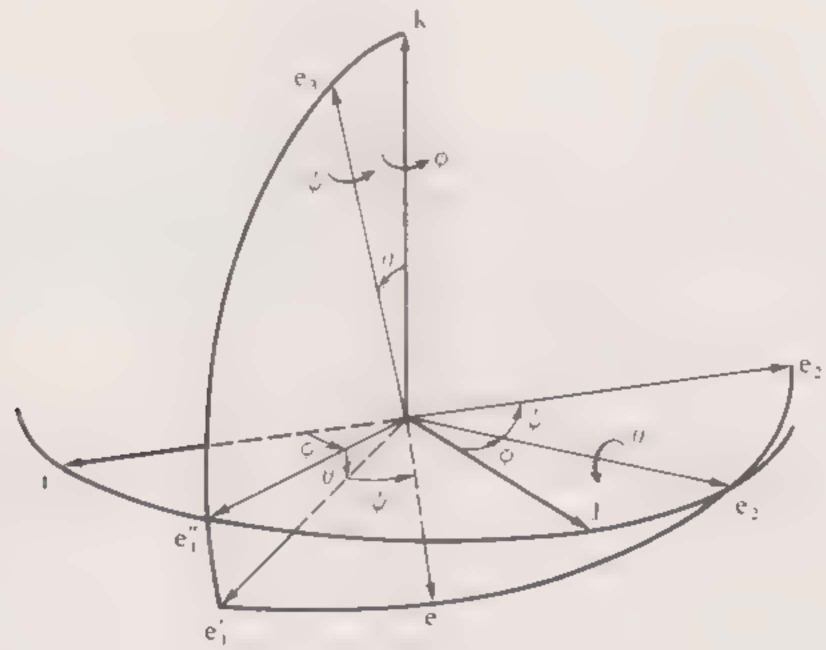


Fig. 10.5

advantage that two of the three terms in (10.19) are already expressed in terms of the axes $\mathbf{e}_1'$, $\mathbf{e}_2'$, $\mathbf{e}_3$.

The components of $\mathbf{k}$ in the directions of $\mathbf{e}_1'$, $\mathbf{e}_2'$, $\mathbf{e}_3$ are clearly $(-\sin\theta, 0, \cos\theta)$, so that

$$\mathbf{k} = -\sin\theta\,\mathbf{e}_1' + \cos\theta\,\mathbf{e}_3. \tag{10.20}$$

Thus, from (10.19), we find

$$\boldsymbol{\omega} = -\dot{\phi}\sin\theta\,\mathbf{e}_1' + \dot{\theta}\mathbf{e}_2' + (\dot{\psi} + \dot{\phi}\cos\theta)\mathbf{e}_3. \tag{10.21}$$

We can now find the angular momentum from (9.21). All we have to do is to multiply the three components of $\boldsymbol{\omega}$ by the principal moments I_1, $\cancel{I_2}$ 1 I_3. We obtain

$$\mathbf{J} = -I_1\dot{\phi}\sin\theta\,\mathbf{e}_1' + I_1\dot{\theta}\mathbf{e}_2' + I_3(\dot{\psi} + \dot{\phi}\cos\theta)\mathbf{e}_3. \tag{10.22}$$

To find equations of motion, we could either re-express **J** in terms of the fixed axes **i**, **j**, **k**, or write down equations directly in the rotating frame defined by $\mathbf{e}_1'$, $\mathbf{e}_2'$, $\mathbf{e}_3$. We shall, however, find a simpler method in the next chapter.

The kinetic energy is obtained similarly from (9.23). It is

$$T = \tfrac{1}{2}I_1\dot{\phi}^2 \sin^2\theta + \tfrac{1}{2}I_1\dot{\theta}^2 + \tfrac{1}{2}I_3(\dot{\psi}+\dot{\phi}\cos\theta)^2. \qquad (10.23)$$

$T = \frac{1}{2}\sum_i I_i \omega_i^2$

10.5 Free Motion of a Symmetric Body

from § 11.3 paragraph 4 p170

As an example of the use of the equations obtained in the preceding section, we consider a symmetric body moving under no forces. Its centre of mass moves of course with uniform velocity. The interesting part of the motion is the rotation about the centre of mass. The same formalism applies to a body rotating freely about a fixed pivot. ?

Since there are no external forces, the angular momentum equation is simply

$$\mathrm{d}\mathbf{J}/\mathrm{d}t = \mathbf{0}.$$

The angular momentum vector therefore points in a fixed direction in space, and its magnitude J is constant. Though we could solve the problem in terms of arbitrary axes, it will be convenient to choose the axis **k** to be in the direction of the constant vector **J**. Then, by (10.20),

$$\mathbf{J} = J\mathbf{k} = -J\sin\theta\,\mathbf{e}_1' + J\cos\theta\,\mathbf{e}_3.$$

For consistency, this must be identically equal to the expression (10.22). Hence, equating the three components, we obtain,

$$\begin{aligned} I_1\dot{\phi} &= J, \\ I_1\dot{\theta} &= 0, \qquad\qquad (10.24) \\ I_3(\dot{\psi}+\dot{\phi}\cos\theta) &= J\cos\theta. \end{aligned}$$

From the second equation, we learn that θ is a constant, and from the other two that $\dot{\phi}$ and $\dot{\psi}$ are constants. Hence the axis of the body, $\mathbf{e}_3$, rotates around the direction of **J** at a constant rate $\dot{\phi}$, maintaining a constant slope, and in addition the body spins about its axis with constant angular velocity $\dot{\psi}$.

The angular velocity vector

$$\boldsymbol{\omega} = -\dot{\phi}\sin\theta\,\mathbf{e}_1' + (\dot{\psi}+\dot{\phi}\cos\theta)\mathbf{e}_3 \qquad (10.25)$$

has fixed components with respect to $\mathbf{e}_1'$, $\mathbf{e}_2'$, $\mathbf{e}_3$. (See Fig. 10.6.) It too therefore maintains a constant angle with $\mathbf{e}_3$, and also with **J**. In space, the angular velocity vector $\boldsymbol{\omega}$ therefore describes a cone

around the direction of J, precessing at the rate $\dot\phi$. This is known as the *space cone*. In the body, $\boldsymbol{\omega}$ maintains a constant angle to the axis $\mathbf{e}_3$, and therefore describes a cone in the body around this axis—the *body cone*. The rate at which it describes the body cone is $\dot\psi$. (For, $\boldsymbol{\omega}$ is fixed with respect to the axes $\mathbf{e}_1'$, $\mathbf{e}_2'$, $\mathbf{e}_3$, and the body rotates relative to them with angular velocity $\dot\psi$.)

Since the direction of $\boldsymbol{\omega}$ is the instantaneous axis of rotation, the body cone is instantaneously at rest along the line in the direction

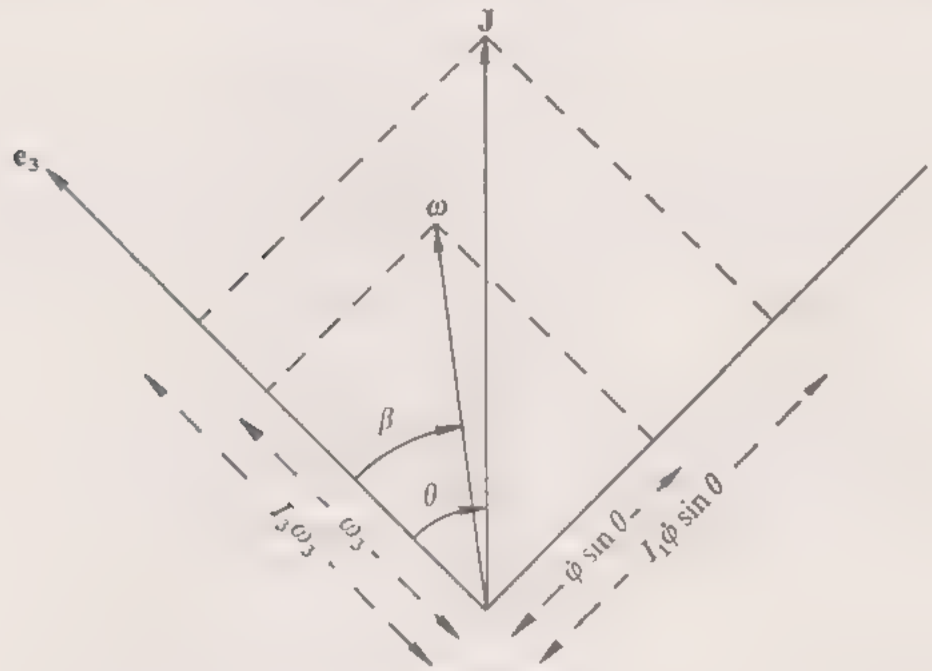


Fig. 10.6

of $\boldsymbol{\omega}$, where it touches the space cone. (See Fig. 10.7.) Thus the motion may be described by saying that the body cone rolls round the fixed space cone.

Eliminating J from the equations (10.24), we find that the angular velocities $\dot\phi$ and $\dot\psi$ are related by

$$\omega_3 = \dot\psi + \dot\phi \cos\theta = \frac{I_1}{I_3}\dot\phi\cos\theta, \tag{10.26}$$

or

$$\dot\psi = \frac{I_1 - I_3}{I_3}\dot\phi\cos\theta. \tag{10.27}$$

The semi-vertical angle β of the body cone is given, from (10.25), by

$$\tan\beta = \frac{\dot\phi\sin\theta}{\dot\psi + \dot\phi\cos\theta} = \frac{I_3}{I_1}\tan\theta. \tag{10.28}$$

We may distinguish two cases. First, if $I_1 > I_3$, that is if the body is *prolate*, the angle β of the body cone is less than θ, and the cones lie outside each other, as shown in Fig. 10.7. On the other hand, if

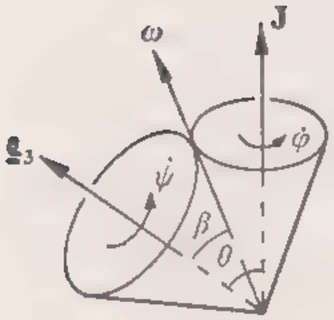


Fig. 10.7

$I_1 < I_3$, which is the case for an *oblate* body, then $\beta > \theta$, and the space cone lies wholly inside the body cone, as in Fig. 10.8. In this case, $\dot{\psi}$ is negative, according to (10.27).

For a body which is totally symmetric ($I_1 = I_3$), the space cone reduces to a line, and the motion of the body is simply a rotation about this axis with angular velocity $\dot{\phi}$. This must, of course, be true, since every axis is in this case a principal axis, so that $\boldsymbol{\omega}$ is parallel to $\mathbf{J}$, and is fixed in magnitude and direction.

For a body which is nearly totally symmetric, so that $I_1 - I_3 \ll I_3$, the angles β and θ are almost equal, and the space cone is much narrower than the body cone. Moreover $\dot{\psi} \ll \dot{\phi}$, so that period taken to describe the body cone is much greater than that for the space cone. The earth's axis has a very small oscillation of this kind, the *Chandler wobble*. It precesses around a fixed direction in space roughly once every day, and moves round a cone in the earth in a period which should be about 300 days if the earth were a perfectly rigid body, but is actually rather longer. (The motion is in any case not as regular as this description would suggest. Note that this precession is quite independent of—and much smaller than—the effect discussed at the end of §10.1.)

10.6 Summary

We have seen that the determination of the motion of a rigid body can be divided into two quite separate problems. The motion of the centre of mass is determined by the total force acting on the body—or, if one point is fixed, the force is determined by the motion of the centre of mass. The rotational motion about the centre of mass is determined by the total moment of the forces acting. In general, to discuss the rotational motion, it is convenient to refer it to a set of principal axes (normally rotating with the body, though in the special case of symmetric bodies, they may be chosen as in §10.5). We shall see in the following chapter that the Lagrangian method is very useful for obtaining equations of motion in terms of the Euler angles.

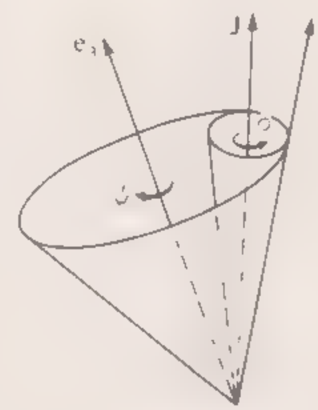


Fig. 10.8

PROBLEMS

1 A gyroscope consisting of a uniform circular disc of mass 100 g and radius 4 cm is pivoted so that its centre of mass is fixed, and is spinning about its axis at 2400 r.p.m. A 5 g mass is attached to the axis at a distance of 10 cm from the centre. Find the angular velocity of precession of the axis.

2 A top consists of a cone of height 5 cm and base radius 2 cm. It is spinning with its vertex fixed at 7200 r.p.m. Find the precessional period of the axis about the vertical. (See Chapter 9, Problem 7 for I_3.)

3 A gyroscope consisting of a uniform solid sphere of radius 10 cm is spinning at 3000 r.p.m. about a horizontal axis. Due to faulty construction, the fixed point is not precisely at the centre of mass, but 0·02 mm away from it on the axis. Find the time taken for the axis to move through 1°.

4 A wheel of radius a is rolling with velocity v round a circle of radius $R(\gg a)$, maintaining a constant inclination α to the vertical. Show that $v = a\omega = R\Omega$, where ω is the angular velocity of the wheel about its axis, and Ω is the precessional angular velocity of the axis. Use the momentum equation to find the horizontal and vertical components of the force at the point of contact. Then show from the angular momentum equation about the centre of mass that $R = 2v^2/g \tan \alpha$. Evaluate R if $v = 5$ m/s and $\alpha = 30°$.

5 A top of spheroidal shape spinning about its axis of symmetry is placed on a rough table. Show, by considering its moment about the centre of mass, that the frictional force at the point of contact will not only make the axis precess, but also *rise* if $c > a$, and *fall* if $c < a$.

6 A solid rectangular box, of dimensions 10 cm × 6 cm × 2 cm, is spinning freely with angular velocity 240 r.p.m. Determine the frequency of small oscillations of the axis if the axis of rotation is (a) the longest, and (b) the shortest, axis.

7 A uniformly charged sphere is spinning freely with angular velocity ω in a uniform magnetic field $\mathbf{B}$. Evaluate the total moment of the magnetic force, choosing appropriate axes, and show that it is equal to $(q/2Mc)\mathbf{J} \wedge \mathbf{B}$. Hence show that the axis will precess around the direction of the magnetic field with angular velocity equal to the angular Larmor frequency of §5.5. What difference would it make if the charge distribution were spherically symmetric, but non-uniform?

8 The average moment exerted by the sun on the earth is, except for sign, identical with the expression found in Chapter 6, Problem 10, provided we interpret m as the mass of the sun, and r as the distance to the sun. Show that $Q = -2(I_3 - I_1)$, and hence that the precessional angular velocity produced by this moment is

$$\boldsymbol{\Omega} = -\frac{3}{2}\frac{I_3 - I_1}{I_3}\frac{\tilde{\omega}^2}{\omega}\cos\alpha\,\mathbf{n},$$

where $\tilde{\omega}$ is the earth's *orbital* angular velocity. Show also that $(I_3 - I_1)/I_3 \approx \varepsilon$, the oblateness of the earth, and hence evaluate $\boldsymbol{\Omega}$. Why is this effect less sensitive to the distribution of density within the earth than the complementary one discussed in §6.5?

9 The axis of a gyroscope is free to rotate within a smooth horizontal circle in colatitude λ. Due to the Coriolis force, there is a couple on the gyroscope. To find the effect of this couple, use the equation for the rate of change of angular momentum in a frame rotating with the earth (e.g. that of Fig. 5.7), $\dot{\mathbf{J}} + \boldsymbol{\Omega} \wedge \mathbf{J} = \mathbf{G}$, where $\mathbf{G}$ is the couple restraining the axis from leaving the horizontal plane, and $\boldsymbol{\Omega}$ is the earth's angular velocity. (Neglect terms of order $\boldsymbol{\Omega}^2$, in particular the contribution of $\boldsymbol{\Omega}$ to $\mathbf{J}$.) From the component along the axis, show that the angular velocity ω about the axis is constant; and from the vertical component show that the angle φ between the axis and east obeys the equation

$$I_1\ddot{\varphi} - I_3\omega\Omega \sin\lambda \cos\varphi = 0.$$

Show that the stable position is with the axis pointing north. Determine the period of small oscillations about this direction if the gyroscope is a flat circular disc spinning at 6000 r.p.m. in latitude 30°N. Explain why this system is sensitive to the horizontal component of $\mathbf{\Omega}$, and describe the effect qualitatively from the point of view of an inertial observer.

Chapter 11 Lagrangian Mechanics

Lagrange's Equations p134

from § 13.2 Conservation of Energy

We have already seen that the equations of motion for a system of N particles moving under conservative forces may be obtained from the Lagrangian function in terms of any set of $3N$ independent co-ordinates. (See §8.6.) In this chapter, we shall give a more systematic account of the Lagrangian method, and apply it in particular to the case of rigid bodies.

11.1 Generalized Co-ordinates; Holonomic Systems

Let us consider a rigid body composed of a large number N of particles. The positions of all the particles may be specified by $3N$ co-ordinates. However, these $3N$ co-ordinates cannot all vary independently, but are subject to constraints—the rigidity conditions. In fact, the position of every particle may be fixed by specifying the values of just six quantities—for instance, the three co-ordinates X, Y, Z of the centre of mass, and the three Euler angles φ, θ, ψ which determine the orientation. These six constitute a set of *generalized co-ordinates* for the rigid body.

These co-ordinates may be subject to further constraints, which may be of two kinds. First, we might for example fix the position of one point of the body. Such constraints are represented by algebraic conditions on the co-ordinates (e.g. $X = Y = Z = 0$), which may be used to eliminate some of the co-ordinates. In this particular case, the three Euler angles alone suffice to fix the position of every particle.

The second type of constraint is represented by conditions on the velocities rather than the co-ordinates. For example, we might constrain the centre of mass to move with uniform velocity, or to move round a circle with uniform angular velocity. Then, in place of algebraic equations, we have differential equations (e.g. $\dot{X} = v$). In simple cases, these equations can be solved to find some of the co-ordinates as explicit functions of time (e.g. $X = X_0 + vt$). Then the position of every particle will be determined by the values of the remaining generalized co-ordinates and the time t.

In general, we say that $q_1, q_2, \ldots, q_n$ is a set of *generalized co-ordinates* for a given system if the position of every particle in the system is a function of these variables, and perhaps also explicitly of time,

$$\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, \ldots, q_n, t). \tag{11.1}$$

The number of co-ordinates which can vary independently is called the number of *degrees of freedom* of the system. If it is possible to

solve the constraint equations, and eliminate some of the co-ordinates, leaving a set equal in number to the number of degrees of freedom, the system is called *holonomic*. If this elimination introduces explicit functions of time, the system is said to be *forced*; on the other hand, if all the constraints are purely algebraic, so that t does not appear explicitly in (11.1), the system is *natural*.

There do exist non-holonomic systems, for which the constraint equations cannot be solved to eliminate some of the co-ordinates. Consider, for example, a sphere rolling on a rough plane. Its position may be specified by five generalized co-ordinates—$X, Y, \varphi, \theta, \psi$. ($Z$ is a constant, and may be omitted.) However, the sphere can only roll in two directions, and the number of degrees of freedom is two (three if we allow it to spin about the vertical). The constraint equations serve to determine the angular velocity in terms of the velocity of the centre of mass. Using the fact that the instantaneous axis of rotation must be a horizontal axis through the point of contact, it is not hard to show that $\omega = a\mathbf{k} \wedge \dot{\mathbf{R}}$. However, these equations cannot be integrated to find the orientation in terms of the position of the centre of mass; for, we could roll the sphere round a circle so that it returns to its starting point but with a different orientation.

It is not particularly difficult to extend Lagrange's method to cover non-holonomic systems of this type, but we shall not consider them here. In what follows, we shall always assume that the constraint equations can be solved, and the number of generalized co-ordinates taken equal to the number of degrees of freedom.

The distinction between a natural and a forced system can be expressed in another way, which will be useful later. Differentiating (11.1) with respect to the time, we find that the velocity of the ith particle is a linear function of $\dot{q}_1, \ldots, \dot{q}_n$, though in general it depends in a more complicated way on the co-ordinates $q_1, \ldots, q_n$ themselves:

$$\dot{\mathbf{r}}_i = \sum_{\alpha=1}^{n} \frac{\partial \mathbf{r}_i}{\partial q_\alpha} \dot{q}_\alpha + \frac{\partial \mathbf{r}}{\partial t}.$$

The last term arises from the explicit dependence on t, and is absent for a natural system. When we substitute in $T = \sum \frac{1}{2} m \dot{\mathbf{r}}^2$, we obtain a quadratic function of the time derivatives $\dot{q}_1, \ldots, \dot{q}_n$. For a natural system, it is a *homogeneous* quadratic function; but for a forced system there are also linear and constant terms.

$T = \frac{1}{2} M \dot{\underline{R}}^2 + T^*$ (8.28)

For example, according to (8.28) and (10.23), the kinetic energy of a symmetric rigid body is

$T = \frac{1}{2} I_1 \dot{\varphi}^2 \sin^2\theta + \frac{1}{2} I_1 \dot{\theta}^2 + \frac{1}{2} I_3 (\dot{\psi} + \dot{\varphi}\cos\theta)^2$ (10.23)

$$T = \tfrac{1}{2}M(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2) + \tfrac{1}{2}I_1^*(\dot{\varphi}^2 \sin^2\theta + \dot{\theta}^2) + \tfrac{1}{2}I_3^*(\dot{\psi} + \dot{\varphi}\cos\theta)^2.$$

If we impose further algebraic constraints, such as $X = 0$, the corresponding terms drop out, and we are still left with a homogeneous quadratic function of the remaining time derivatives. On the other hand, if we impose differential constraints, such as $\dot{X} = v$, or $\dot{\phi} = \omega$, we obtain a function with constant or linear terms.

11.2 Lagrange's Equations

To obtain a more general form of Lagrange's equations than the one found in §3.6, let us return for the moment to the case of a single particle. We suppose that it is moving under an arbitrary force F, and consider variations of the integral

$$I = \int_{t_0}^{t_1} T \, \mathrm{d}t, \tag{11.2}$$

where $T = \frac{1}{2}m\dot{\mathbf{r}}^2$.

Now let us make a small variation $\delta x(t)$ in the x co-ordinate, subject to the conditions $\delta x(t_0) = \delta x(t_1) = 0$. Then clearly $\delta T = m\dot{x}\,\delta\dot{x}$. Substituting in (11.2), and performing an integration by parts, in which the integrated term vanishes, we find

$$\delta I = -\int_{t_0}^{t_1} m\ddot{x}\,\delta x \, \mathrm{d}t.$$

Now the integrand is $-F_x\delta x = -\delta W$, where δW is the work done by the force $\mathbf{F}$ in the displacement δx. Similarly, if we make variations of all three co-ordinates, we find

$$\delta I = -\int_{t_0}^{t_1} \delta W \, \mathrm{d}t, \tag{11.3}$$

where $\delta W = \mathbf{F}\cdot\delta\mathbf{r}$ is the work done by the force in the displacement $\delta\mathbf{r}$.

The advantage of (11.3) is that, like Hamilton's principle, it makes no explicit reference to any particular set of co-ordinates. Hence we can use it to find equations of motion in terms of an arbitrary set of co-ordinates q_1, q_2, q_3. Let us define the generalized forces F_1, F_2, F_3 corresponding to $\mathbf{F}$ by

$$\delta W = F_1\,\delta q_1 + F_2\,\delta q_2 + F_3\,\delta q_3. \tag{11.4}$$

$$\delta I = \int_{x_0}^{x_1} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right] \delta y(x) \, dx$$

Then we may equate δI as given by (11.3) to the general expression (3.31). Consider, for example, a variation δq_1 of q_1. Then, according to (3.31),

$$\delta I = \int_{t_0}^{t_1} \left[\frac{\partial T}{\partial q_1} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_1}\right)\right] \delta q_1(t) \, \mathrm{d}t.$$

On the other hand, by (11.3) and (11.4),

$$\delta I = -\int_{t_0}^{t_1} F_1\, \delta q_1(t)\, \mathrm{d}t.$$

These two expressions must be equal for arbitrary variations subject only to the condition that $\delta q_1(t_0) = \delta q_1(t_1) = 0$. Hence the integrands must be equal, and we obtain *Lagrange's equations* in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) = \frac{\partial T}{\partial q_\alpha} + F_\alpha. \tag{11.5}$$

For example, for a particle moving in a plane, and described by polar co-ordinates, $T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$. Hence Lagrange's equations are

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{r}) &= mr\dot{\theta}^2 + F_r, \\ \frac{\mathrm{d}}{\mathrm{d}t}(mr^2\dot{\theta}) &= F_\theta. \end{aligned} \tag{11.6}$$

It is important to remember that F_θ here does *not* mean the component of the vector $\mathbf{F}$ in the direction of increasing θ. (Compare the discussion following (3.38).) It is defined by (11.4), or → p 46

$$\delta W = F_r\, \delta r + F_\theta\, \delta\theta.$$

The work done by a force in producing a small angular displacement $\delta\theta$ is equal to $\delta\theta$ multiplied by the moment of the force about the axis of rotation. Thus F_θ is in fact the moment of $\mathbf{F}$ about the origin. (Recall that $p_\theta = mr^2\dot{\theta}$ is the angular momentum. Thus (11.6) expresses the fact that the rate of change of angular momentum is equal to the moment of the force.)

If the force $\mathbf{F}$ is conservative, then $\delta W = -\delta V$, where V is the potential energy. It then follows from (11.4) that

$$F_\alpha = -\frac{\partial V}{\partial q_\alpha}. \tag{11.7}$$

Thus, defining the Lagrangian function $L = T - V$, we can write (11.5) in the form (3.38) obtained earlier,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) = \frac{\partial L}{\partial q_\alpha}. \tag{11.8}$$

Note that the 'generalized force' $\partial L/\partial q_\alpha$ on the right side of this equation includes both the force $\partial T/\partial q_\alpha$ arising from the curvilinear

nature of the co-ordinates (for example, the centrifugal term $mr\dot{\theta}^2$ in (11.6)), and the force F_α determined by $\mathbf{F}$.

More generally, even if $\mathbf{F}$ is non-conservative, it may be possible to find a 'potential energy' V depending on the velocities as well as the co-ordinates, and such that

$$F_\alpha = -\frac{\partial V}{\partial q_\alpha} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial V}{\partial \dot{q}_\alpha}\right). \tag{11.9}$$

In this case too, we can define $L = T - V$, and write Lagrange's equations in the form (11.8). The most important example of a force of this type is the electromagnetic force on a charged particle, discussed in §11.5. Note that in this case, the 'potential energy' contributes not only to the generalized force on the right side of (11.8), but also to the generalized momentum on the left.

There is nothing in the discussion leading to Lagrange's equations (11.5) or (11.8) which restricts its applicability to the case of a single particle. The general principle embodied in (11.3) applies to any system whatsoever, containing any number of particles. If the system is holonomic, with n degrees of freedom, then we can make independent arbitrary variations of the generalized co-ordinates $q_1, q_2, \ldots, q_n$, and obtain Lagrange's equations (11.5), in which the generalized forces are defined, as in (11.4), by

$$\delta W = \sum_{\alpha=1}^{n} F_\alpha\, \delta q_\alpha. \tag{11.10}$$

Finally, if the forces are conservative, or more generally if they can be written in the form (11.9), then Lagrange's equations may be expressed in the form (11.8).

11.3 Precession of a Symmetric Top

As a first example of the use of Lagrange's equations, we consider in more detail the problem of the symmetric top, discussed in the previous chapter. This is a symmetric rigid body, pivoted at a point on its axis of symmetry, and moving under gravity. (Compare Fig. 10.3.) This system has three degrees of freedom, and we may use the three Euler angles as generalized co-ordinates. The kinetic energy is given by (10.23), and the potential energy is

$$V = Mgr\cos\theta.$$

Thus the Lagrangian function is

$$L = \tfrac{1}{2}I_1\dot{\varphi}^2\sin^2\theta + \tfrac{1}{2}I_1\dot{\theta}^2 + \tfrac{1}{2}I_3(\dot{\psi} + \dot{\varphi}\cos\theta)^2 - Mgr\cos\theta. \tag{11.11}$$

Lagrange's equation for θ,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta}$$

is therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}(I_1\dot{\theta}) = I_1\dot{\varphi}^2 \sin\theta\cos\theta - I_3(\dot{\psi}+\dot{\varphi}\cos\theta)\dot{\varphi}\sin\theta + Mgr\sin\theta. \tag{11.12}$$

The Lagrangian function does not involve the other two Euler angles, but only their time derivatives, and the corresponding equations express the constancy of the generalized momenta p_φ and p_ψ,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}[I_1\dot{\varphi}\sin^2\theta + I_3(\dot{\psi}+\dot{\varphi}\cos\theta)\cos\theta] &= 0, \\ \frac{\mathrm{d}}{\mathrm{d}t}[I_3(\dot{\psi}+\dot{\varphi}\cos\theta)] &= 0. \end{aligned} \tag{11.13}$$

2 Note that this last equation tells us that the component of angular velocity ω_3 about the axis of symmetry is constant,

$$\omega_3 = \dot{\psi}+\dot{\varphi}\cos\theta = \text{constant}. \tag{11.14}$$

3 We now discuss the conditions under which steady precession is possible. We look for solutions of the equations of motion in which the angle of inclination of the axis, θ, remains constant. From the equations (11.13), we learn that in this case both $\dot{\varphi}$ and $\dot{\psi}$ must be constant. Thus the axis of the top precesses around the vertical with constant angular velocity $\dot{\varphi} = \Omega$. To find the relation between the angular velocities Ω and ω_3, we examine the θ equation (11.12). Since the left side must vanish, we obtain (for $\sin\theta \neq 0$, that is unless the axis is vertical)

$$I_1\Omega^2\cos\theta - I_3\omega_3\Omega + Mgr = 0. \tag{11.15}$$

4 Thus, for a given inclination, and a given value of ω_3, there are two possible precessional angular velocities. It is instructive to examine the case where ω_3 is large. Then there is one root for Ω which is much less than ω_3, given approximately by

$$\Omega \approx \frac{Mgr}{I_3\omega_3},$$

and another which is comparable in magnitude to ω_3,

$$\Omega \approx \frac{I_3\omega_3}{I_1\cos\theta}.$$

The first of these is just the approximate result we obtained in §10.1 on the assumption that Ω was much smaller than ω_3. The second represents a rapid precessional motion in which the gravitational force is negligible. It corresponds precisely to the free precessional motion of a rigid body, discussed in §10.5. (Compare (10.5) and (10.26).)

For smaller values of ω_3, these approximate solutions are not adequate, and we must use the exact roots of (11.15). For any given inclination $\theta < \frac{1}{2}\pi$, there is a minimum value of ω_3 for which steady precession can occur, given by

$$I_3^2\omega_3^2 = 4I_1 Mgr\cos\theta.$$

If the top is spinning more slowly than this, it will start to wobble. If we allow θ to be greater than $\frac{1}{2}\pi$—that is, if we consider a top suspended below its point of support—then steady precession is possible for any value of ω_3. In particular, for $\omega_3 = 0$, we find the possible angular velocities of a compound pendulum swinging in a circle,

$$\Omega = \pm\left(\frac{Mgr}{I_1|\cos\theta|}\right)^{1/2}.$$

We shall discuss the more general motion of this system in Chapter 13, using the Hamiltonian methods developed there.

11.4 Pendulum Constrained to Rotate about an Axis

Next, let us consider the system illustrated in Fig. 11.1. It consists of a light rigid rod of length l, carrying a mass m at one end, and hinged at the other end to a vertical axis, so that it can swing freely in a vertical plane. We suppose first that, in addition to the force of gravity, a known torque G is applied to the axis.

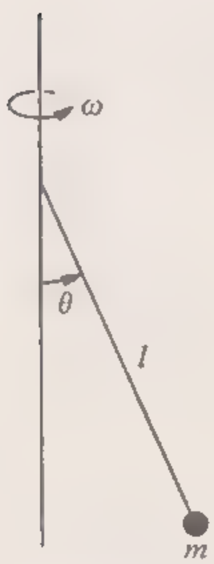


Fig. 11.1

This system has two degrees of freedom. Its position may be described by the two polar angles θ, φ. (For convenience, we take the polar axis vertically downward, so that $\theta = 0$ is the equilibrium position.)

The kinetic energy of the system is

$$T = \tfrac{1}{2}ml^2(\dot\theta^2 + \dot\varphi^2\sin^2\theta),$$

and the gravitational potential energy is

$$V = mgl(1-\cos\theta).$$

These may be combined in the Lagrangian function

$$L = \tfrac{1}{2}ml^2(\dot\theta^2 + \dot\varphi^2\sin^2\theta) - mgl(1-\cos\theta). \qquad (11.16)$$

Since the work done by the torque G is $\delta W = G\delta\varphi$, Lagrange's equations are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot\theta}\right) = \frac{\partial L}{\partial \theta}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot\varphi}\right) = \frac{\partial L}{\partial \varphi} + G,$$

or, explicitly,

$$ml^2\ddot\theta = ml^2\dot\varphi^2 \sin\theta\cos\theta - mgl\sin\theta, \tag{11.17}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}(ml^2\dot\varphi\sin^2\theta) = G. \tag{11.18}$$

4) Now let us suppose that the torque G is adjusted to constrain the system to rotate with constant angular velocity ω about the vertical. This imposes the constraint $\dot\varphi = \omega$, and the system may then be regarded as a system with one degree of freedom, described by the co-ordinate θ. Substituting this constraint in the Lagrangian function (11.16), we obtain

$$L = \tfrac{1}{2}ml^2(\dot\theta^2 + \omega^2\sin^2\theta) - mgl(1-\cos\theta). \tag{11.19}$$

Note the appearance in the kinetic energy part of L of a term independent of $\dot\theta$, which is characteristic of a forced system.

5) It is often useful in problems of this kind to separate the Lagrangian function into a 'kinetic energy' term T', quadratic in the time derivatives, and a 'potential energy' term $-V'$, independent of them. We write $L = T' - V'$, where

$$T' = \tfrac{1}{2}ml^2\dot\theta^2, \qquad V' = mgl(1-\cos\theta) - \tfrac{1}{2}ml^2\omega^2\sin^2\theta. \quad (11.19a)$$

Physically, this corresponds to using a rotating frame, rotating with angular velocity ω. T' is the kinetic energy of the motion relative to this frame, and the extra term in V' is the potential energy corresponding to the centrifugal force.

6) For a forced system, the constraining forces can do work on the system, and the total energy $T+V$ is not in general a constant. However, in our case, there is still a conservation law. Multiplying (11.17) by $\dot\theta$ and integrating, we find

$$T' + V' = E' = \text{constant}. \tag{11.20}$$

This is not equal to $T+V$, because the centrifugal term appears with the opposite sign.

It is easy to verify the consistency of this result by calculating the work done on the system. The rate at which the torque G does work is $G\omega$. Thus from (11.18)

$$\frac{\mathrm{d}}{\mathrm{d}t}(T+V) = G\omega = \frac{\mathrm{d}}{\mathrm{d}t}(ml^2\omega^2\sin^2\theta).$$

Since $T'+V' = T+V-ml^2\omega^2\sin^2\theta$, this is equivalent to (11.20).

We can use this conservation law, just as we did in §2.1, to find the qualitative features of the motion. To do this, we draw the 'potential energy' diagram for the function V'. There are two distinct cases, depending on the value of ω. If $\omega^2 < g/l$ (that is, if the rotation period is longer than the free oscillation period of the pendulum), then V' has a minimum at $\theta = 0$, and a maximum at $\theta = \pi$, as shown in Fig. 11.2. The motion is qualitatively just like that of an ordinary pendulum, though the period is longer.

If $\omega^2 > g/l$, then V' has maxima at both 0 and π, and an intermediate minimum at $\cos\theta = g/l\omega^2$. (See Fig. 11.3.) The equilibrium position at $\theta = 0$ is then unstable. The stable equilibrium position is the one where the transverse components of the gravitational and centrifugal forces are in equilibrium. For this case, three types of motion are possible. If $E' < 0$, the pendulum oscillates around this stable position, without ever reaching the vertical. For $0 < E' < 2mgl$, it swings from one side to the other as before, though $\theta = 0$ is no longer the position of maximum velocity. Finally, if $E' > 2mgl$, we have a continuous circular motion.

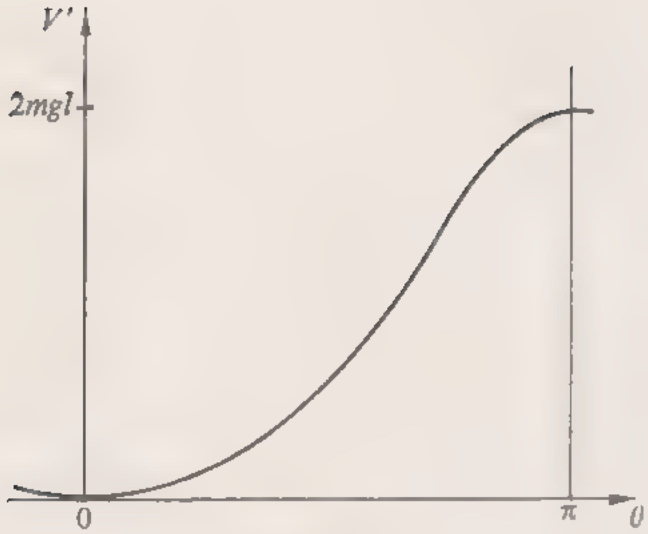


Fig. 11.2

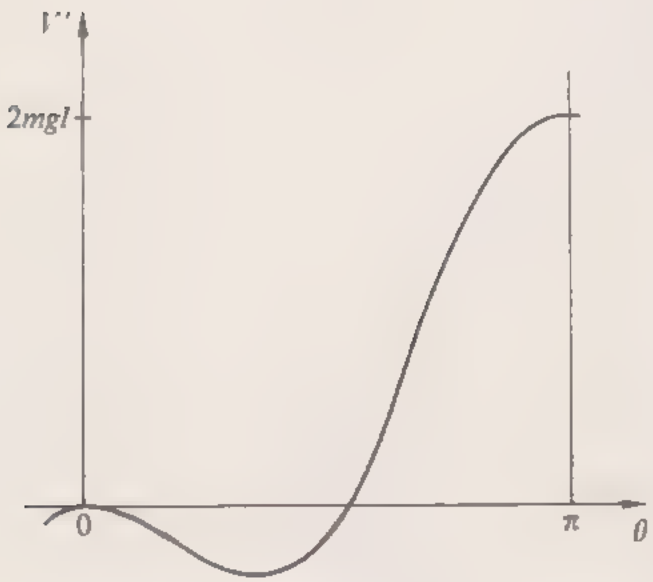


Fig. 11.3

11.5 Charged Particle in an Electromagnetic Field see notes on p 168.

We now wish to consider one of the most important examples of a non-conservative force. We consider a particle of charge q moving in an electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$. The force on the particle is then*

$$\mathbf{F} = q\left(\mathbf{E} + \frac{1}{c}\dot{\mathbf{r}} \wedge \mathbf{B}\right), \tag{11.21}$$

or, in terms of components,

$$F_x = qE_x + \frac{q}{c}(\dot{y}B_z - \dot{z}B_y), \tag{11.22}$$

with two similar equations obtained by cyclic permutation of x, y, z.

Now we wish to show that this force may be written in the form (11.9) with a suitably chosen function V. To do this, we have to make use of a standard result of electromagnetic theory (see Appendix B), according to which it is always possible to find a 'scalar potential' ϕ, and a 'vector potential' $\mathbf{A}$, functions of $\mathbf{r}$ and t such that

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \wedge \mathbf{A}. \tag{11.23}$$

For time-independent fields, the scalar potential ϕ is simply the electrostatic potential of Chapter 6.

* In SI units, the factor of c here and in what follows should be deleted.

Now let us consider the function

$$V = q\phi(\mathbf{r}, t) - \frac{q}{c}\,\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r}, t)$$

$$= q\phi - \frac{q}{c}(\dot{x}A_x + \dot{y}A_y + \dot{z}A_z). \qquad (11.24)$$

Clearly,

$$-\frac{\partial V}{\partial x} = -q\frac{\partial\phi}{\partial x} + \frac{q}{c}\left(\dot{x}\frac{\partial A_x}{\partial x} + \dot{y}\frac{\partial A_y}{\partial x} + \dot{z}\frac{\partial A_z}{\partial x}\right). \qquad (11\text{-}24b)$$

Also,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial V}{\partial \dot{x}}\right) = -\frac{q}{c}\frac{\mathrm{d}A_x}{\mathrm{d}t} \qquad (11\text{-}24c)$$

$$= -\frac{q}{c}\left(\frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_x}{\partial y}\dot{y} + \frac{\partial A_x}{\partial z}\dot{z}\right), \qquad (11\text{-}24d)$$

since A_x varies with time both because of its explicit time dependence, and because of its dependence on the particle position $\mathbf{r}$. Hence, adding, we obtain

$$-\frac{\partial V}{\partial x} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial V}{\partial \dot{x}}\right) = q\left(-\frac{\partial\phi}{\partial x} - \frac{1}{c}\frac{\partial A_x}{\partial t}\right)$$

$$+\frac{q}{c}\left[\dot{y}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) + \dot{z}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\right] \quad (11\text{-}24e)$$

$$= qE_x + \frac{q}{c}(\dot{y}B_z - \dot{z}B_y) \qquad (11\text{-}24f)$$

by (11.23). But this is just the expression for F_x given by (11.22). Hence we have verified that $\mathbf{F}$ is given by (11.9) with V equal to (11.24).

It follows that the equations of motion for a particle in an electromagnetic field may be obtained from the Lagrangian function

$$L = \tfrac{1}{2}m\dot{\mathbf{r}}^2 + \frac{q}{c}\,\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r}, t) - q\phi(\mathbf{r}, t). \qquad (11.25)$$

Note the appearance in L of terms linear in the time derivatives. This function cannot be separated into two parts, one quadratic in $\dot{\mathbf{r}}$, and one independent of it. Another consequence of the appearance of these terms is that the generalized momentum p_x is no longer equal to the familiar mechanical momentum $m\dot{x}$. Instead,

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} + \frac{q}{c}A_x, \qquad (11\text{-}25b)$$

or, more generally,

$$\mathbf{p} = m\dot{\mathbf{r}} + \frac{q}{c}\mathbf{A}. \tag{11.26}$$

We can now obtain the equations of motion in terms of arbitrary co-ordinates from the Lagrangian function (11.25). For example, in terms of cylindrical polars, it reads

$$L = \tfrac{1}{2}m(\dot{\rho}^2+\rho^2\dot{\varphi}^2+\dot{z}^2)+\frac{q}{c}(\dot{\rho}A_\rho+\rho\dot{\varphi}A_\varphi+\dot{z}A_z)-q\phi. \tag{11.27}$$

Let us consider, as a simple example, the case of a uniform static magnetic field **B**. In this case, we may take $\phi = 0$, and $\mathbf{A} = \frac{1}{2}\mathbf{B}\wedge\mathbf{r}$, or, if **B** is in the z direction,

$$A_\rho = 0, \qquad A_\varphi = \tfrac{1}{2}B\rho, \qquad A_z = 0. \qquad (11.27b)$$

(It is easy to verify the relation $\mathbf{B} = \nabla\wedge\mathbf{A}$ using (A.47).) Thus the Lagrangian function is

$$L = \tfrac{1}{2}m(\dot{\rho}^2+\rho^2\dot{\varphi}^2+\dot{z}^2)+\frac{qB}{2c}\rho^2\dot{\varphi}. \tag{11.28}$$

Lagrange's equations are, therefore,

$$m\ddot{\rho} = m\rho\dot{\varphi}^2+\frac{qB}{c}\rho\dot{\varphi}, \qquad (11.28b)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m\rho^2\dot{\varphi}+\frac{qB}{2c}\rho^2\right) = 0, \tag{11.29}$$

$$m\ddot{z} = 0. \qquad (11.29b)$$

In particular, let us find the solutions in which ρ is constant. In that case, we learn from the last two equations that $\dot{\varphi}$ and $\dot{z}$ are also constants, and from the first equation that either $\dot{\varphi} = 0$ (particle moving parallel to the z-axis) or

(5-12) $$\dot{\varphi} = -\frac{qB}{mc}. \qquad (11.29c)$$

This is of course precisely the solution we obtained in §5.2.

The second of the three equations (11.29) is particularly interesting. It shows that although in general the z component of the particle angular momentum J_z is not a constant, there is still a corresponding conservation law for the quantity

$$p_\varphi = m\rho^2\dot{\varphi}+\frac{qB}{2c}\rho^2.$$

The reason for the existence of such a conservation law will be discussed in Chapter 13.

11.6 The Stretched String

As a final example of the use of the Lagrangian method, we consider a rather different kind of problem. This is an example of a system with an *infinite* number of degrees of freedom—a string of length l, and mass μ per unit length, with fixed ends, and stretched to a tension F. Referred to from § 12·6 Normal Modes of a Stretched String.

We shall consider small transverse oscillations of the string. In place of a finite set of generalized co-ordinates, we now have a continuous function, the displacement $y(x, t)$ of the string from its equilibrium position. The kinetic energy of a small element of length dx is $\frac{1}{2}(\mu dx)\dot{y}^2$, where of course $\dot{y} = \partial y/\partial t$. Thus the total kinetic energy is

$$T = \int_0^l \tfrac{1}{2}\mu\dot{y}^2 \, dx.$$

When the string is in equilibrium, its length is l. However, when it is displaced, its length is given by (3.28), and is § 3·5 The Calculus of Variations

$$l+\Delta l = \int_0^l (1+y'^2)^{1/2} \, dx, \qquad (3{\cdot}28)$$

where $y' = \partial y/\partial x$. The work done against the tension in increasing the length by Δl is $F\Delta l$. This is the potential energy of the string. For small displacements, we can replace $(1+y'^2)^{1/2}$ by $1+\frac{1}{2}y'^2$. Thus

$$V = \int_0^l \tfrac{1}{2}Fy'^2 \, dx. \tag{11.30}$$

Our Lagrangian function is, therefore,

$$L = \int_0^l (\tfrac{1}{2}\mu\dot{y}^2 - \tfrac{1}{2}Fy'^2) \, dx. \tag{11.31}$$

It has the form

$$L = \int_0^l \mathscr{L}(y, \dot{y}, y') \, dx, \tag{11.32}$$

where $\mathscr{L}$ (in our case independent of y) may be called the Lagrangian density.

We can use Hamilton's principle as before to find the equations of motion corresponding to the Lagrangian (11.32). The action integral is

$$I = \int_{t_0}^{t_1} \int_0^l \mathscr{L}(y, \dot{y}, y') \, dx \, dt.$$

Let us consider a small variation $\delta y(x, t)$, which vanishes at t_0 and t_1, and also (because the ends of the string are fixed) at $x = 0$ and $x = l$. The variation of the action integral is

$$\delta I = \int_{t_0}^{t_1} \int_0^l \left[\frac{\partial \mathscr{L}}{\partial y} \delta y + \frac{\partial \mathscr{L}}{\partial \dot{y}} \frac{\partial}{\partial t}(\delta y) + \frac{\partial \mathscr{L}}{\partial y'} \frac{\partial}{\partial x}(\delta y) \right] \mathrm{d}x \, \mathrm{d}t.$$

Now, as in §3.5, we integrate by parts, with respect to t in the second term, and with respect to x in the third. In both cases, the integrated term vanishes because δy is zero at the limits of integration. Thus we obtain

$$\delta I = \int_{t_0}^{t_1} \int_0^l \left[\frac{\partial \mathscr{L}}{\partial y} - \frac{\partial}{\partial t}\left(\frac{\partial \mathscr{L}}{\partial \dot{y}}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \mathscr{L}}{\partial y'}\right) \right] \delta y(x, t) \, \mathrm{d}x \, \mathrm{d}t.$$

Hamilton's principle requires that this expression should vanish for arbitrary variations $\delta y(x, t)$ vanishing at the limits. This is possible only if the integrand vanishes identically. Hence we obtain Lagrange's equations

$$\frac{\partial \mathscr{L}}{\partial y} - \frac{\partial}{\partial t}\left(\frac{\partial \mathscr{L}}{\partial \dot{y}}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \mathscr{L}}{\partial y'}\right) = 0. \tag{11.33}$$

For the Lagrangian (11.31),

$$\frac{\partial \mathscr{L}}{\partial y} = 0, \qquad \frac{\partial \mathscr{L}}{\partial \dot{y}} = \mu \dot{y}, \qquad \frac{\partial \mathscr{L}}{\partial y'} = -Fy'.$$

Thus Lagrange's equation becomes

$$\ddot{y} = c^2 y'', \tag{11.34}$$

where

$$c^2 = F/\mu. \tag{11.35}$$

Note that c^2 has the dimensions of (velocity)^2. In fact, as we show below, c is the velocity of propagation of a wave moving along the string. Equation (11.34) is the one-dimensional *wave equation*. Similar equations occur in many branches of physics, whenever wave phenomena are encountered.

The general solution of (11.34) involves two arbitrary *functions*, which may be determined by the initial values of y and $\dot{y}$. It is easy to verify by direct substitution that, for any function f,

$$y = f(x - ct)$$

is a solution. It represents a wave travelling along the string with velocity c to the right; for, the shape of the displacement y is the

same at time t as at time 0, but is shifted to the right by a distance ct. (See Fig. 11.4.) Similarly,

$$y = f(x+ct)$$

is a solution, representing a wave travelling to the left along the string. The general solution is

$$y = f(x+ct)+g(x-ct).$$

To satisfy the boundary conditions at $x = 0$ and $x = l$, both terms in the solution must be present. For a string of finite length,

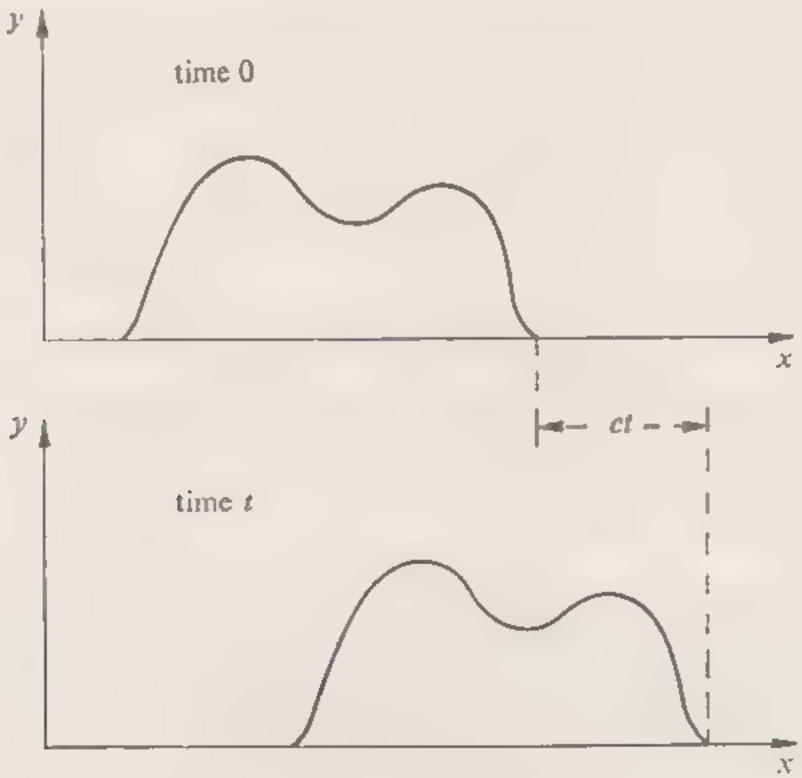


Fig. 11.4

a wave cannot travel indefinitely in one direction. When it gets to the end of the string, it must be reflected back. The condition $y =$ 0 at $x = 0$ requires

$$f(ct)+g(-ct) = 0.$$

Thus the functions f and g differ only in sign, and in the sign of the argument, and we can write the solution as

$$y = f(ct+x)-f(ct-x). \tag{11.36}$$?

The condition $y = 0$ at $x = l$ requires

$$f(ct+l) = f(ct-l).$$

In other words, f must be a *periodic* function, with period $2l$,

$$f(x+2l) = f(x). \tag{11.37}$$

To establish the connection between f and the initial conditions, let us write f as the sum of an even function and an odd function,

$$\begin{aligned} f(x) &= \tfrac{1}{2}[h(x)+k(x)], \\ h(x) &= f(x)+f(-x), \\ k(x) &= f(x)-f(-x). \end{aligned} \tag{11.38}$$

To fix f, we have to specify its values in the interval from $-l$ to l, or, equivalently, to specify both h and k in the interval from 0 to l. From (11.36), the initial value of y is

$$y(x, 0) = f(x)-f(-x) = k(x). \tag{11.39}$$

This therefore determines the odd part of f. Moreover, the initial value of $\dot{y}$ is

$$\dot{y}(x, 0) = cf'(x)-cf'(-x) = ch'(x). \tag{11.40}$$

It determines the even part of f (up to an irrelevant additive constant, which cancels in (11.36)).

11.7 Summary

The position of every part of a system may be fixed by specifying the values of a set of generalized co-ordinates. If these co-ordinates can all vary independently, the system is holonomic. This is the case in all the examples we have considered. The system is natural if the functions specifying the positions of particles in terms of the generalized co-ordinates do not involve the time explicitly. In that case, the kinetic energy is a homogeneous quadratic function of the $\dot{q}_\alpha$. For a forced system, on the other hand, T may contain linear and constant terms. In either case, the equations of motion are given by Lagrange's equations. If the forces are conservative (and sometimes in other cases too) all we need is the Lagrangian function, $L = T-V$. In general, for dissipative forces, the generalized forces F_α corresponding to the generalized co-ordinates q_α must be found by evaluating the work done in a small displacement.

PROBLEMS

1 Evaluate accurately the two precessional angular velocities of the top described in Chapter 10, Problem 2, if the axis makes an angle of 30° with

the vertical. Find also the minimum angular velocity ω_3 for which steady precession at this angle is possible.

2 A uniform solid cylindrical drum of mass M and radius a is free to rotate about its axis, which is horizontal. A cable of negligible mass and length l is wound on the drum, and carries on its free end a mass m. Write down the Lagrangian function in terms of an appropriate generalized co-ordinate, assuming no slipping or stretching of the cable. If the cable is initially fully wound up, and the system is released from rest, find the angular velocity of the drum when it is fully unwound.

3 By treating the length l of the cable in the preceding question as variable, and finding Lagrange's equations for two generalized co-ordinates, show that the tension in the cable is $Mmg/(M+2m)$.

4 Find the Lagrangian function for the system of Problem 2 if the cable is elastic with elastic potential energy $\frac{1}{2}kx^2$, where x is the extension of the cable. Show that the motion of the mass m is a uniform acceleration at the same rate as before, with a superimposed oscillation of angular frequency given by $\omega^2 = k(M+2m)/Mm$. Find the amplitude of this oscillation if the system is released from rest with the cable unextended.

5 Write down the kinetic energy of a particle in cylindrical polar co-ordinates in a frame rotating with angular velocity ω about the z-axis. Show that the terms proportional to ω and ω^2 reproduce the Coriolis force and the centrifugal force respectively.

6 Show that the kinetic energy of the gyroscope described in Chapter 10, Problem 9, is

$$T = \tfrac{1}{2}I_1(\Omega \sin\lambda \cos\varphi)^2 + \tfrac{1}{2}I_1(\dot{\varphi}+\Omega\cos\lambda)^2 + \tfrac{1}{2}I_3(\dot{\psi}+\Omega\sin\lambda\sin\varphi)^2.$$

From Lagrange's equations, show that the angular velocity ω_3 about the axis is constant, and obtain the equation for φ without neglecting Ω^2. Show that motion with the axis pointing north becomes unstable for very small values of ω_3, and find the smallest value for which it is stable. What are the stable positions when $\omega_3 = 0$? Interpret this result in terms of a non-rotating frame.

7 Find the Lagrangian function for a symmetric top whose pivot is free to slide on a smooth horizontal table, in terms of the generalized co-ordinates X, Y, φ, θ, ψ, and the principal moments about the centre of mass. (Note that Z is related to θ.) Show that the horizontal motion of the centre of mass may be completely separated from the rotational motion. What difference is there in the equation (11.15) for steady precession? Are the precessional angular velocities greater or less than in the case of a fixed pivot? Show that steady precession at a given value of θ can occur for a smaller value of ω_3 than in the case of a fixed pivot.

8 Use Hamilton's principle to show that if F is any function of the generalized co-ordinates, then the Lagrangian functions L and $L+\mathrm{d}F/\mathrm{d}t$ must yield the same equations of motion. Hence show that the equations of motion of a charged particle in an electromagnetic field are unaffected by the 'gauge transformation' (B.8). (Take $F = -(q/c)\Lambda$.)

9. The stretched string of §11.6 is released from rest with its mid-point displaced a distance a, and each half of the string straight. Find the function $f(x)$. Describe the shape of the string after (a) a short time t, (b) a time $l/2c$, (c) a time l/c.

Chapter 12 Small Oscillations and Normal Modes

In this chapter, we shall discuss a generalization of the harmonic oscillator problem treated in §2.2—the oscillations of a system of several degrees of freedom near a position of equilibrium. We consider only conservative, holonomic systems, described by n generalized co-ordinates $q_1, q_2, \ldots, q_n$. Without loss of generality, we may choose the position of equilibrium to be $q_1 = q_2 = \ldots = q_n = 0$. We shall begin by investigating the form of the kinetic and potential energy functions near this point.

12.1 Orthogonal Co-ordinates

We shall restrict our considerations to natural systems, for which the kinetic energy is a homogeneous quadratic function of $\dot{q}_1, \dot{q}_2 \ldots, \dot{q}_n$. (More generally, we could include also those forced systems—like that of §11.4—for which L can be written as a sum of a quadratic term T' and a term $-V'$ independent of the time derivatives. The only essential restriction is that there should be no linear terms.)

For example, for $n = 2$, we have

$$T = \tfrac{1}{2}a_{11}\dot{q}_1{}^2 + a_{12}\dot{q}_1\dot{q}_2 + \tfrac{1}{2}a_{22}\dot{q}_2{}^2. \tag{12.1}$$

In general, the coefficients a_{11}, a_{12}, a_{22} will be functions of q_1 and q_2. However, for sufficiently small values of q_1 and q_2, we may neglect this dependence, and treat them as constants.

For a particle described by curvilinear co-ordinates, the co-ordinates are called *orthogonal* if the co-ordinate curves always intersect at right angles. In that case, the kinetic energy contains terms in $\dot{q}_1{}^2$, $\dot{q}_2{}^2$, $\dot{q}_3{}^2$, but no cross products like $\dot{q}_1\dot{q}_2$. By an extension of this terminology, the generalized co-ordinates $q_1, q_2, \ldots, q_n$ are called *orthogonal* if T is a sum of squares with no cross products—for example if a_{12} above is zero.

It is a considerable simplification to choose the co-ordinates to be orthogonal, and this can always be done. For instance, we may set

$$q_1' = q_1 + \frac{a_{12}}{a_{11}} q_2,$$

so that, in terms of q_1' and q_2, (12.1) becomes

$$T = \tfrac{1}{2}a_{11}\dot{q}_1'^2 + \tfrac{1}{2}a_{22}'\dot{q}_2{}^2, \tag{12.2}$$

with

$$a'_{22} = a_{22} - \frac{a_{12}^{\ 2}}{a_{11}}.$$

We can even go further. Since T is necessarily positive, the coefficients in (12.2) must be positive numbers. Hence we can define new co-ordinates $q''_1 = (a_{11})^{1/2} q'_1$ and $q''_2 = (a'_{22})^{1/2} q_2$, so that T is reduced to

$$T = \tfrac{1}{2}\dot{q}''^{2}_1 + \tfrac{1}{2}\dot{q}''^{2}_2. \qquad (12.3)$$

A similar procedure may be used in the general case. We may first eliminate the cross products involving q_1 by means of the transformation to

$$q'_1 = q_1 + \frac{a_{12}}{a_{11}} q_2 + \ldots + \frac{a_{1n}}{a_{11}} q_n,$$

then those involving q_2, and so on. Thus we can always reduce T to the standard form

$$T = \sum_{\alpha=1}^{n} \tfrac{1}{2}\dot{q}_\alpha^{\ 2}. \qquad (12.4)$$

As an illustration of these ideas, let us consider the double pendulum illustrated in Fig. 12.1. It consists of a pendulum of mass M and length L with a second pendulum of mass m and length l suspended from it. We consider only motion in a plane, so that the system has two degrees of freedom. As generalized co-ordinates, we may choose the inclinations θ and φ to the downward vertical.

The velocity of the upper pendulum bob is $L\dot{\theta}$. That of the lower bob has two components—the velocity $L\dot{\theta}$ of its point of support, and the velocity $l\dot{\varphi}$ relative to that point. The angle between these is $\varphi - \theta$. Hence the kinetic energy is

$$T = \tfrac{1}{2}ML^2\dot{\theta}^2 + \tfrac{1}{2}m[L^2\dot{\theta}^2 + l^2\dot{\varphi}^2 + 2Ll\dot{\theta}\dot{\varphi}\cos(\varphi - \theta)].$$

For small values of θ and φ, we may approximate $\cos(\varphi - \theta)$ by 1. Since there is a term in $\dot{\theta}\dot{\varphi}$, these co-ordinates are not orthogonal, but we can make them orthogonal by adding an appropriate multiple of θ to φ. In fact, it is easy to see that a pair of orthogonal co-ordinates is provided by the displacements

$$x = L\theta, \qquad y = L\theta + l\varphi, \qquad (12.5)$$

in terms of which T becomes

$$T = \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}m\dot{y}^2. \qquad (12.6)$$

We could, if we wished, complete the reduction to the standard form (12.4) by writing $q_1 = M^{1/2}x$, $q_2 = m^{1/2}y$. In practice, this is

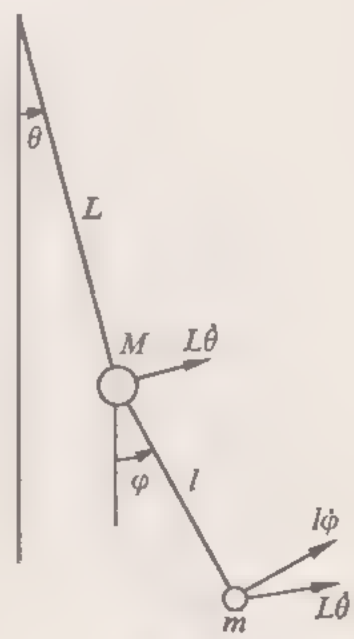


Fig. 12.1

not always essential, though it makes the discussion of the general case much easier.

12.2 Equations of Motion for Small Oscillations

Now let us consider the potential energy function V. With T given by (12.4), the equations of motion are simply

$$\ddot{q}_\alpha = -\frac{\partial V}{\partial q_\alpha}. \tag{12.7}$$

Thus the condition for equilibrium is that all n partial derivatives of V should vanish at the equilibrium position. For small values of the co-ordinates, we can expand V in a series. For example, for $n = 2$,

$$V = V_0+(b_1q_1+b_2q_2)+(\tfrac{1}{2}k_{11}q_1{}^2+k_{12}q_1q_2+\tfrac{1}{2}k_{22}q_2{}^2)+\dots.$$

The equilibrium conditions require that the linear terms should be zero, $b_1 = b_2 = 0$, just as in §2.2. Moreover, the constant term V_0 is arbitrary, and may be set equal to zero. Thus the leading terms are the quadratic ones, and for small values of q_1 and q_2 we may approximate V by

$$V = \tfrac{1}{2}k_{11}q_1{}^2+k_{12}q_1q_2+\tfrac{1}{2}k_{22}q_2{}^2. \tag{12.8}$$

Then the equations of motion (12.7) become

$$\begin{aligned} \ddot{q}_1 &= -k_{11}q_1-k_{12}q_2, \\ \ddot{q}_2 &= -k_{21}q_1-k_{22}q_2, \end{aligned} \tag{12.9}$$

where for the sake of symmetry we have written $k_{21}=k_{12}$.

In the general case, V may be taken to be a homogeneous quadratic function of the co-ordinates, which can be written

$$V = \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\tfrac{1}{2}k_{\alpha\beta}q_\alpha q_\beta, \tag{12.10}$$

with $k_{\beta\alpha} = k_{\alpha\beta}$. Then the equations of motion are

$$\ddot{q}_\alpha = -\sum_{\beta=1}^{n}k_{\alpha\beta}q_\beta. \tag{12.11}$$

For example, in the case of the double pendulum, the potential energy is

$$V = (M+m)gL(1-\cos\theta)+mgl(1-\cos\varphi).$$

For small angles, we can approximate $1-\cos\theta$ by $\frac{1}{2}\theta^2$. Hence in terms of the orthogonal co-ordinates (12.5) we obtain

$$V = \frac{(M+m)g}{2L}x^2+\frac{mg}{2l}(y-x)^2.$$

Thus the equations of motion are

$$\ddot{x} = -\left[\frac{(M+m)g}{ML}+\frac{mg}{Ml}\right]x+\frac{mg}{Ml}y,$$
$$\ddot{y} = \frac{g}{l}x-\frac{g}{l}y. \tag{12.12}$$

Note the inequality of the coefficients corresponding to k_{12} and k_{21} of (21.9). This arises from the fact that we have not absorbed the normalization factors $M^{1/2}$ and $m^{1/2}$ into x and y.

12.3 Normal Modes

The general solution of the pair of second-order differential equations (12.9) must involve four arbitrary constants, which may be fixed by the initial values of $q_1, q_2, \dot{q}_1, \dot{q}_2$. Similarly, the general solution of (12.11) must involve $2n$ arbitrary constants. To find this general solution, we look first for solutions in which all the co-ordinates are oscillating with the same frequency ω,

$$q_\alpha = A_\alpha e^{i\omega t}, \tag{12.13}$$

where the A_α are complex constants. (As in §2.3, the physical solution is represented by the real part of (12.13).) Such solutions are called *normal modes* of oscillation of the system.

Substituting (12.13) into (12.11), we obtain a set of n simultaneous linear equations for the n amplitudes A_α,

$$-\omega^2 A_\alpha+\sum_{\beta=1}^{n} k_{\alpha\beta}A_\beta = 0. \tag{12.14}$$

Let us consider first the case $n = 2$. Then these equations are

$$(k_{11}-\omega^2)A_1+k_{12}A_2 = 0,$$
$$k_{21}A_1+(k_{22}-\omega^2)A_2 = 0. \tag{12.15}$$

Referred to from Appendix C §C.3. and given explanation as example on page 239.

These equations are only mutually consistent if

$$(k_{11}-\omega^2)(k_{22}-\omega^2)-k_{12}{}^2 = 0. \tag{12.16}$$

This is called the *characteristic equation* for the system. It determines the frequencies of the normal modes. As a quadratic equation for ω^2, its discriminant may be written $(k_{11}-k_{22})^2+4k_{12}{}^2$, which is clearly positive. Thus its roots are always real.

The condition for stability is that both roots should be positive. A negative root would yield a solution of the form (12.13) with real

exponents, corresponding to a exponential increase in the displacements with time.

If ω^2 is chosen equal to one of the two roots, then either of the equations (12.15) determines the ratio A_1/A_2. Since the coefficients are real numbers, this ratio is obviously real. This means that A_1 and A_2 have the same phase (or phases differing by π), so that q_1 and q_2 not only oscillate with the same frequency, but actually in phase (except perhaps for sign). The ratio of q_1 to q_2 remains fixed throughout the motion.

There remains in A_1 and A_2 a common arbitrary complex factor, which serves to fix the overall amplitude and phase of the normal mode solution. Thus each normal mode solution contains two arbitrary real constants. Since the equations of motion (12.9) are linear, any linear superposition of solutions is again a solution. Hence the general solution is simply a superposition of the two normal mode solutions. If ω^2 and ω'^2 are the roots of (12.16), it may be written as the real part of

$$\begin{aligned} q_1 &= A_1 e^{i\omega t} + A_1' e^{i\omega' t}, \\ q_2 &= A_2 e^{i\omega t} + A_2' e^{i\omega' t}, \end{aligned} \tag{12.17}$$

in which the ratios A_1/A_2 and A'_1/A'_2 are fixed by (12.15).

In the case of the double pendulum, the equations (12.15) are

$$\begin{aligned} \left[\frac{(M+m)g}{ML}+\frac{mg}{Ml}-\omega^2\right]A_x - \frac{mg}{Ml}A_y &= 0, \\ -\frac{g}{l}A_x + \left[\frac{g}{l}-\omega^2\right]A_y &= 0. \end{aligned} \tag{12.18}$$

The characteristic equation (12.16) simplifies to

$$\omega^4 - \frac{M+m}{M}\left(\frac{g}{L}+\frac{g}{l}\right)\omega^2 + \frac{M+m}{M}\frac{g^2}{Ll} = 0. \tag{12.19}$$

The roots of this equation determine the frequencies of the two normal modes.

It is interesting to examine certain special cases. First, let us suppose that the upper pendulum is very heavy ($M \gg m$). Then the two roots, with the corresponding ratios determined by (12.18), are, approximately,

$$\omega^2 \approx \frac{g}{l}, \qquad \frac{A_x}{A_y} \approx \frac{m}{M}\frac{L}{l-L},$$

and

$$\omega^2 \approx \frac{g}{L}, \qquad \frac{A_x}{A_y} \approx \frac{L-l}{L}.$$

In the first mode, the upper pendulum is practically stationary, while the lower one is swinging with its natural frequency. In the second mode, whose frequency is that of the upper pendulum, the amplitudes are of comparable magnitude.

At the other extreme, if $M \ll m$, the two normal modes are

$$\omega^2 \approx \frac{g}{L+l}, \qquad \frac{A_x}{A_y} \approx \frac{L}{L+l},$$

and

$$\omega^2 \approx \frac{m}{M}\left(\frac{g}{L}+\frac{g}{l}\right), \qquad \frac{A_x}{A_y} \approx -\frac{m}{M}\frac{L+l}{L}.$$

In the first mode, the pendulums swing like a single rigid pendulum of length $L+l$. In the second, the lower bob remains almost stationary, while the upper one executes a very rapid oscillation.

The normal modes of a system with n degrees of freedom may be found by a very similar method. The condition for consistency of the simultaneous equations (12.14) is that the determinant of the coefficients should vanish. For example, for $n = 3$, we require

$$\begin{vmatrix} k_{11}-\omega^2 & k_{12} & k_{13} \\ k_{21} & k_{22}-\omega^2 & k_{23} \\ k_{31} & k_{32} & k_{33}-\omega^2 \end{vmatrix} = 0. \tag{12.20}$$

This is a cubic equation for ω^2. It can be proved that its three roots are all real.* As before, the condition for stability is that all three roots should be positive. The roots then determine the frequencies of the three normal modes.

For each normal mode, the ratios of the amplitudes are fixed by the equations (12.14). As in the case $n = 2$, they are all real, so that in a normal mode the co-ordinates oscillate in phase. Each normal mode solution involves just two arbitrary constants, and the general solution is a superposition of all the normal modes.

12.4 Coupled Oscillators

One often encounters examples of physical systems which may be described as two (or more) harmonic oscillators, which are approximately independent, but with some kind of relatively weak coupling between the two. (As a specific example, we shall consider below the

* This is a consequence of the symmetry of the determinant (12.20) under reflection in the leading diagonal. Essentially this result is proved, in a different context, in Appendix C.

system shown in Fig. 12.2, which consists of a pair of identical pendulums coupled by a spring.)

If the co-ordinate q of a harmonic oscillator is normalized so that $T = \frac{1}{2}\dot{q}^2$, then $V = \frac{1}{2}\omega^2 q^2$, where ω is the angular frequency. Hence for a pair of uncoupled oscillators, the coefficients in (12.8) are $k_{11} = \omega_1^2$, $k_{12} = 0$ and $k_{22} = \omega_2^2$. When the oscillators are weakly coupled these equalities will still be approximately true, so that in particular k_{12} is small in comparison to k_{11} and k_{22}. Thus from (12.16) it is clear that the characteristic frequencies of the coupled system are given by $\omega^2 \approx k_{11}$ and $\omega^2 \approx k_{22}$. As one might expect, they are close to the frequencies of the uncoupled oscillators. Then from (12.15) we see that, in the first normal mode, the ratio A_2/A_1 is approximately $k_{12}/(k_{11}-k_{22})$. Thus, *unless* the frequencies of the two oscillators are nearly equal, the normal modes differ very little from those of the uncoupled system, and the coupling is not of great importance. The interesting case, in which it can be important, is that in which the frequencies are equal, or nearly so.

As a specific example of this case, we consider a pair of pendulums, each of mass m and length l, coupled by a weak spring. (See Fig. 12.2.) We shall use the displacements x and y as generalized co-ordinates. Then, in the absence of coupling, the potential energy is approximately $\frac{1}{2}m\omega_0^2(x^2+y^2)$, where $\omega_0^2 = g/l$ gives the free oscillation frequency. The potential energy of the spring has the form $\frac{1}{2}k(x-y)^2$. It will be convenient to introduce another frequency ω_s defined by $\omega_s^2 = k/m$. In fact, ω_s is the angular frequency of oscillation of the spring if one end is fixed and the other attached to a mass m. Thus we take

$$V = \tfrac{1}{2}m(\omega_0^2+\omega_s^2)(x^2+y^2)-m\omega_s^2 xy. \tag{12.21}$$

The normal mode equations (12.15) now read

$$\begin{aligned}(\omega_0^2+\omega_s^2-\omega^2)A_x-\omega_s^2 A_y &= 0,\\ -\omega_s^2 A_x+(\omega_0^2+\omega_s^2-\omega^2)A_y &= 0.\end{aligned} \tag{12.22}$$

The two solutions of the characteristic equation are easily seen to be $\omega^2 = \omega_0^2$ (with $A_x/A_y = 1$) and $\omega^2 = \omega_0^2+2\omega_s^2$ (with $A_x/A_y = -1$).

In the first normal mode, the two pendulums oscillate together with equal amplitude. (See Fig. 12.3.) Since the spring is neither expanded nor contracted in this motion, it is not surprising that the frequency is just that of the uncoupled pendulums. In the second normal mode, which has a higher frequency, the pendulums swing in opposite directions, alternately expanding and compressing the spring. (See Fig. 12.4.)

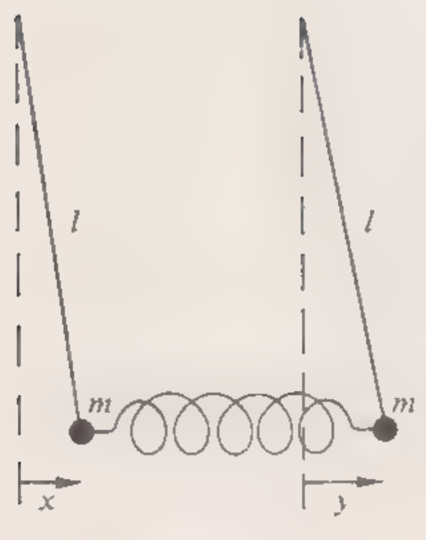


Fig. 12.2

The general solution is a superposition of these two normal modes, and may be written as the real part of

$$\begin{aligned} x &= Ae^{i\omega_0 t}+A'e^{i\omega' t}, \\ y &= Ae^{i\omega_0 t}-A'e^{i\omega' t} \end{aligned} \tag{12.23}$$

where $\omega'^2 = \omega_0^2+2\omega_s^2$. The constants A and A' may be determined by the initial conditions. For example, if the system is released

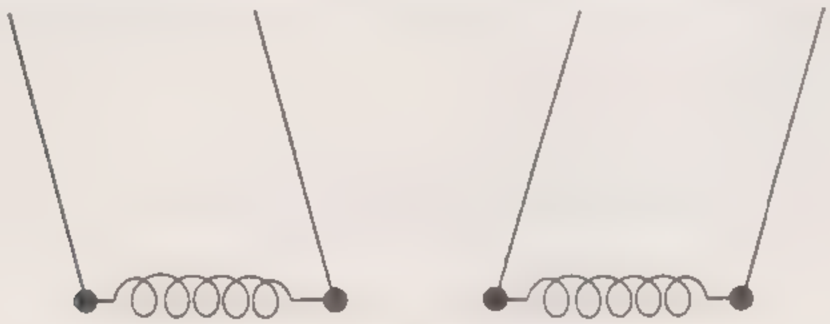


Fig. 12.3

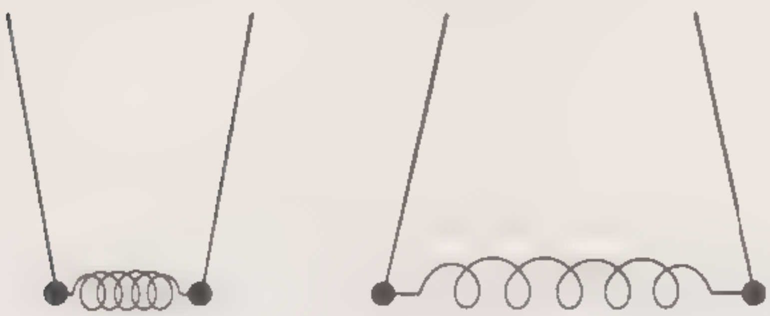


Fig. 12.4

from rest with one pendulum displaced a distance a from its equilibrium position, so that at $t=0$ we have $x=a$, $y=0$, $\dot{x}=\dot{y}=0$, then we easily find that the solution is

$$x = \tfrac{1}{2}a\cos\omega_0 t+\tfrac{1}{2}a\cos\omega' t,$$

$$y = \tfrac{1}{2}a\cos\omega_0 t-\tfrac{1}{2}a\cos\omega' t.$$

Using standard trigonometrical identities, it may be written

$$x = a\cos\omega_- t\cos\omega_+ t, \qquad y = a\sin\omega_- t\sin\omega_+ t,$$

where $\omega_\pm = \frac{1}{2}(\omega'\pm\omega_0)$.

Since the spring is weak, ω' is only slightly greater than ω_0, and therefore $\omega_- \ll \omega_+$. Thus we may describe the motion as follows. The first pendulum swings with angular frequency ω_+, and gradually decreasing amplitude $a\cos\omega_- t$. Meanwhile, the second pendulum

starts to swing with the same frequency, but 90° out of phase, and with gradually increasing amplitude $a \sin \omega_- t$. After a time $\pi/2\omega_-$, the first pendulum has come momentarily to rest, and the second is oscillating with amplitude a. The whole process is then repeated indefinitely (though in practice there will of course be some damping).

This behaviour should be contrasted with that of a pair of coupled oscillators of very different frequencies. In such a case, if one is started oscillating, one of the two normal modes will have a much larger amplitude than the other. Thus only a very small oscillation will be set up in the second oscillator, and the amplitude of the first will be practically constant.

Normal Co-ordinates. The two normal modes of this (or any) system are completely independent. We can make this fact explicit by introducing a new pair of co-ordinates in place of x and y. Let us set

$$q_1 = \frac{m^{1/2}}{\sqrt{2}}(x+y), \qquad q_2 = \frac{m^{1/2}}{\sqrt{2}}(x-y).$$

In terms of these co-ordinates, the solution (12.23) is

$$\begin{aligned} q_1 &= A_1 e^{i\omega_0 t}, \qquad A_1 = (2m)^{1/2} A, \\ q_2 &= A_2 e^{i\omega' t}, \qquad A_2 = (2m)^{1/2} A'. \end{aligned} \tag{12.24}$$

Thus in each normal mode one co-ordinate only is oscillating. Co-ordinates with this property are called *normal co-ordinates.*

The independence of the two normal co-ordinates may also be seen by examining the Lagrangian function. In terms of q_1 and q_2, the kinetic energy is $T = \frac{1}{2}\dot{q}_1{}^2 + \frac{1}{2}\dot{q}_2{}^2$, while the potential energy function (12.21) is $\frac{1}{2}\omega_0{}^2(q_1{}^2+q_2{}^2)+\omega_s{}^2 q_2{}^2$. Thus

$$L = \tfrac{1}{2}(\dot{q}_1{}^2 - \omega_0{}^2 q_1{}^2) + \tfrac{1}{2}(\dot{q}_2{}^2 - \omega'^2 q_2{}^2). \tag{12.25}$$

In effect, we have reduced the Lagrangian to that for a pair of uncoupled oscillators with angular frequencies ω_0 and ω'.

The normal co-ordinates are very useful in studying the effect on the system of a prescribed external force. For example, suppose that the first pendulum is subjected to a periodic force $F(t) = F_1 e^{i\omega_1 t}$. To find the equations of motion in the presence of this force, we must evaluate the work done in a small displacement. This is

$$F(t)\,\delta x = \frac{F(t)}{(2m)^{1/2}}(\delta q_1 + \delta q_2).$$

Thus the equations of motion are

$$\ddot{q}_1 = -\omega_0{}^2 q_1 + \frac{F_1 e^{i\omega_1 t}}{(2m)^{1/2}},$$
$$\ddot{q}_2 = -\omega'^2 q_2 + \frac{F_1 e^{i\omega_1 t}}{(2m)^{1/2}}. \qquad (12.26)$$

These independent oscillator equations may be solved as in §2.6. In particular, the amplitudes of the forced oscillations are given by

$$A_1 = \frac{F_1/(2m)^{1/2}}{\omega_0{}^2 - \omega_1{}^2}, \qquad A_2 = \frac{F_1/(2m)^{1/2}}{\omega'^2 - \omega_1{}^2}.$$

(Of course, we should really include the effect of damping, so that the amplitudes do not become infinite at resonance.) Note that if the forcing frequency is very close to ω_0, the first normal mode will predominate, and the pendulums will swing in the same direction; while if it is close to ω' the second mode will be more important.

12.5 Oscillations of Particles on a String

Consider a light string of length $(n+1)l$, stretched to a tension F, with n equal masses m spaced along it at regular intervals l.

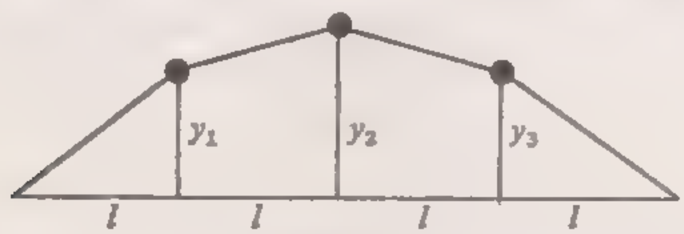


Fig. 12.5

We shall consider transverse oscillations of the particles, and use as our generalized co-ordinates the displacements $y_1, y_2, \ldots, y_n$. (See Fig. 12.5.) Since the kinetic energy is

$$T = \tfrac{1}{2}m(\dot{y}_1{}^2 + \dot{y}_2{}^2 + \ldots + \dot{y}_n{}^2), \qquad (12.27)$$

these co-ordinates are orthogonal.

Next, we must calculate the potential energy. Let us consider the length of string between the jth and $(j+1)$th particles. In equilibrium, its length is l, but when the particles are displaced it is

$$l + \delta l = [l^2 + (y_{j+1} - y_j)^2]^{1/2}$$
$$\approx l\left[1 + \frac{(y_{j+1} - y_j)^2}{2l^2}\right].$$

$V = \int_0^l \frac{1}{2} F y'^2 \, dx$

This applies also to the sections of the string at each end if we set $y_0 = y_{n+1} = 0$. The work done against the tension in increasing the length of the string by this amount is $F\delta l$. Thus, adding the contributions from each piece of the string, we find that the potential energy is

$$V = \frac{F}{2l}[y_1{}^2+(y_2-y_1)^2+\ldots+(y_n-y_{n-1})^2+y_n{}^2]. \quad (12.28)$$

It is worth noting that the potential energy of a continuous string may be obtained as a limiting case, as $n\to\infty$ and $l\to 0$. For small l, $(y_{j+1}-y_j)^2/l^2$ is approximately y'^2, so that we recover the expression (11.30).

From (12.27) and (12.28), we find that Lagrange's equations are

$$\begin{aligned} \ddot{y}_1 &= \frac{F}{ml}(-2y_1+y_2), \\ \ddot{y}_2 &= \frac{F}{ml}(y_1-2y_2+y_3), \\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \ddot{y}_n &= \frac{F}{ml}(y_{n-1}-2y_n). \end{aligned} \quad (12.29)$$

It will be convenient to write $\omega_0{}^2 = F/ml$. Then, substituting the normal mode solution $y_j = A_j e^{i\omega t}$, we obtain the equations

$$\begin{aligned} (2\omega_0{}^2-\omega^2)A_1-\omega_0{}^2A_2 &= 0, \\ -\omega_0{}^2A_1+(2\omega_0{}^2-\omega^2)A_2-\omega_0{}^2A_3 &= 0, \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot& \\ -\omega_0{}^2A_{n-1}+(2\omega_0{}^2-\omega^2)A_n &= 0. \end{aligned} \quad (12.30)$$

For $n = 1$, there is of course just one normal mode, with $\omega^2 = 2\omega_0{}^2$. For $n = 2$, the characteristic equation is

$$(2\omega_0{}^2-\omega^2)^2-\omega_0{}^4 = 0,$$

and we obtain two normal modes with

$$\omega^2 = \omega_0{}^2, \qquad A_1/A_2 = 1,$$

and

$$\omega^2 = 3\omega_0{}^2, \qquad A_1/A_2 = -1.$$

Now let us examine the case $n = 3$. The characteristic equation is now

$$\begin{vmatrix} 2\omega_0{}^2-\omega^2 & -\omega_0{}^2 & 0 \\ -\omega_0{}^2 & 2\omega_0{}^2-\omega^2 & -\omega_0{}^2 \\ 0 & -\omega_0{}^2 & 2\omega_0{}^2-\omega^2 \end{vmatrix} = 0. \quad (12.31)$$

Expanding this determinant by the usual rules, we obtain a cubic equation for ω^2,

$$(2\omega_0{}^2-\omega^2)^3-2\omega_0{}^4(2\omega_0{}^2-\omega^2) = 0.$$

The roots of this equation are $2\omega_0{}^2$ and $(2\pm\sqrt{2})\omega_0{}^2$. Hence we obtain three normal modes

$$\begin{aligned}
\omega^2 &= (2-\sqrt{2})\omega_0{}^2, & A_1: A_2: A_3 &= 1: \sqrt{2}: 1;\\
\omega^2 &= 2\omega_0{}^2, & A_1: A_2: A_3 &= 1: 0: -1;\\
\omega^2 &= (2+\sqrt{2})\omega_0{}^2, & A_1: A_2: A_3 &= 1: -\sqrt{2}: 1.
\end{aligned}$$

These normal modes are illustrated in Fig. 12.6.

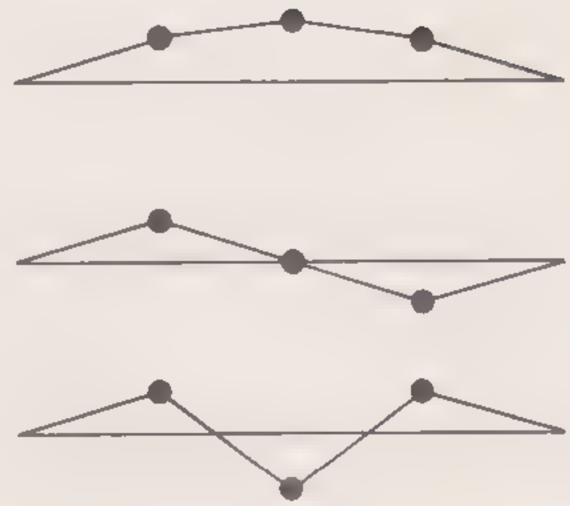


Fig. 12.6

Higher values of n may be treated similarly. For $n = 4$, for example, the characteristic equation is given by the vanishing of a 4×4 determinant, which may be expanded by similar rules to yield

$$(2\omega_0{}^2-\omega^2)^4-3\omega_0{}^4(2\omega_0{}^2-\omega^2)^2+\omega_0{}^8 = 0.$$

The roots of this equation are given by

$$(2\omega_0{}^2-\omega^2)^2 = \frac{3\pm\sqrt{5}}{2}\,\omega_0{}^4 = \left(\frac{\sqrt{5}\pm 1}{2}\,\omega_0{}^2\right)^2.$$

Thus we obtain four normal modes:

$$\begin{aligned}
\omega^2 &= 0{\cdot}38\omega_0{}^2, & A_1: A_2: A_3: A_4 &= 1: 1{\cdot}62: 1{\cdot}62: 1;\\
\omega^2 &= 1{\cdot}38\omega_0{}^2, & A_1: A_2: A_3: A_4 &= 1{\cdot}62: 1: -1: -1{\cdot}62;\\
\omega^2 &= 2{\cdot}62\omega_0{}^2, & A_1: A_2: A_3: A_4 &= 1{\cdot}62: -1: -1: 1{\cdot}62;\\
\omega^2 &= 3{\cdot}62\omega_0{}^2, & A_1: A_2: A_3: A_4 &= 1: -1{\cdot}62: 1{\cdot}62: -1.
\end{aligned}$$

(See Fig. 12.7.)

For every value of n, the slowest mode is the one in which all the masses are oscillating in the same direction, while the fastest is one in which alternate masses oscillate in opposite directions. For large values of n, the normal modes approach those of a continuous stretched string, which we discuss in the following section.

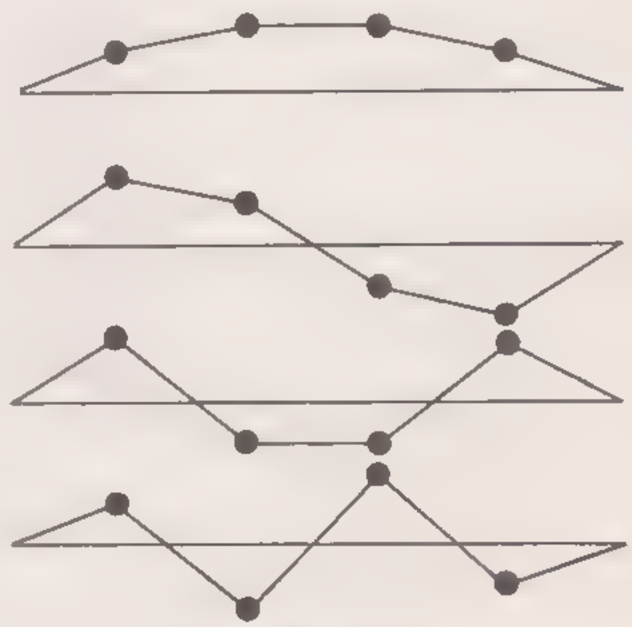


Fig. 12.7

12.6 Normal Modes of a Stretched String

We now wish to discuss the problem treated in §11.6 from the point of view of normal modes. We start from the equation of motion (11.34),

$$\ddot{y} = c^2 y'', \qquad c^2 = F/\mu, \tag{12.32}$$

and look for normal mode solutions of the form

$$y(x, t) = A(x)e^{i\omega t}. \tag{12.33}$$

Substituting in (12.32), we obtain

$$A''(x)+k^2A(x) = 0, \qquad k = \omega/c.$$

Thus, in place of a set of simultaneous equations for the amplitudes A_j, we obtain a differential equation for the amplitude function $A(x)$.

The general solution of this equation is

$$A(x) = a\cos kx + b\sin kx.$$

However, because the ends of the string are fixed, we must impose the boundary conditions $A(0) = A(l) = 0$. Thus $a = 0$, and $\sin kl = 0$. The possible values of k are

$$k = \frac{n\pi}{l}, \qquad n = 1, 2, 3, \dots . \tag{12.34}$$

Each of these values of k corresponds to a normal mode of the string. The corresponding angular frequencies are

$$\omega = \frac{n\pi c}{l}, \quad n = 1, 2, 3, \dots . \tag{12.35}$$

(11-36) written as
$y = f(ct+x) - f(ct-x)$
(11-37) states
$f(x+2l) = f(x)$

They are all multiples of the *fundamental* frequency $\pi c/l = \pi(F/Ml)^{1/2}$ where M is the total mass of the string.

The solution for the nth normal mode can be written as the real part of

$$y(x, t) = A_n e^{in\pi ct/l}\sin\frac{n\pi x}{l}, \tag{12.36}$$

where A_n is an arbitrary complex constant. It represents a 'standing wave' of wave-length $2l/n$, with $n-1$ *nodes*, or points at which $y = 0$. The first few normal modes are illustrated in Fig. 12.8.

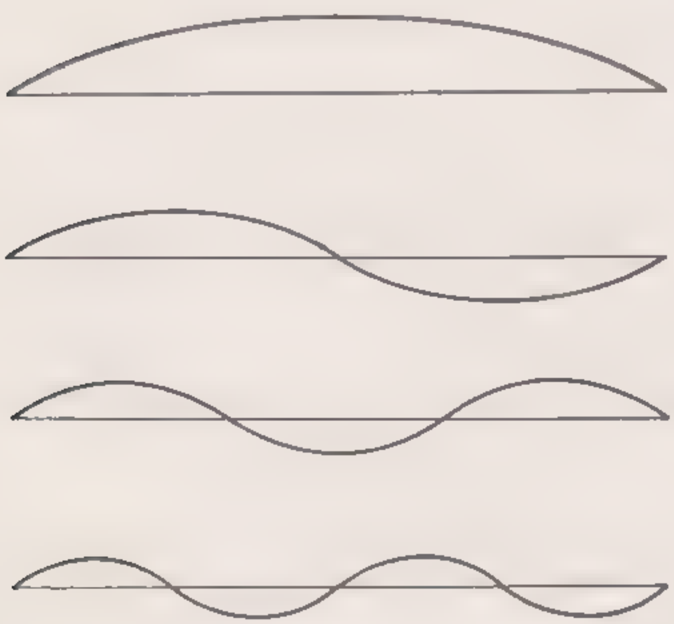


Fig. 12.8

The general solution for the stretched string is a superposition of all the normal modes (12.36). It is easy to establish the connection with the general solution (11.36) obtained in the previous chapter. According to (11.37), $f(x)$ is a periodic function, with period $2l$. Hence it may be expanded in a Fourier series (see equation (2.41))

$$f(x) = \sum_{n=-\infty}^{+\infty} f_n e^{in\pi x/l}.$$

$f(x) \to f(t \pm l)$
$f(t \pm l) \to F(t)$
$\to (2.41)$

Thus the solution (11.36) is

$$\begin{aligned} y(x, t) &= \sum_{n=-\infty}^{+\infty} f_n[e^{in\pi(ct+x)/l} - e^{in\pi(ct-x)/l}] \\ &= \sum_{n=-\infty}^{+\infty} 2if_n e^{in\pi ct/l}\sin\frac{n\pi x}{l} \\ &= 2\,\mathrm{Re}\sum_{n=1}^{+\infty} A_n e^{in\pi ct/l}\sin\frac{n\pi x}{l}, \end{aligned}$$

with $A_n = 2if_n$. (Recall that f_n and f_{-n} must be complex conjugates.)

12.7 Summary

Near a position of equilibrium of any natural, conservative system, the kinetic energy T may be taken to be a homogeneous quadratic function of the $\dot{q}_\alpha$, with constant coefficients, and the potential energy V to be a homogeneous quadratic function of the q_α. We can always find a set of orthogonal co-ordinates, in terms of which T is reduced to a sum of squares. Lagrange's equations then take on a simple form. To find the normal modes of oscillation, we substitute solutions of the form $q_\alpha = A_\alpha e^{i\omega t}$, and obtain a set of simultaneous linear equations for the coefficients. The condition for consistency of these equations is the characteristic equation, which determines the frequencies of the normal modes. The stability condition is that all the roots of this equation for ω^2 should be positive.

The problem of finding the normal modes is equivalent to that of finding normal co-ordinates, which reduce not only T but also V to a sum of squares. In terms of the normal co-ordinates, the system is reduced to a set of uncoupled harmonic oscillators, whose frequencies are the characteristic frequencies of the system. The general solution to the equations of motion is a superposition of all the normal modes. In it, each normal co-ordinate is oscillating at its own frequency, and with amplitude and phase determined by the initial conditions.

PROBLEMS

1 Find the normal modes of a pair of coupled pendulums (like those of Fig. 12.2) if the two pendulums are of different masses M and m. If the pendulum of mass M is started oscillating with amplitude a, find the maximum amplitude of the other pendulum in the subsequent oscillation. Does the amplitude of the first pendulum ever fall to zero?

2 Three identical springs, of negligible mass, spring constant k, and natural length a, are attached end-to-end, and a pair of particles, each of mass m, are fixed to the points where they meet. The system is stretched between fixed points a distance $3l$ apart. Find the frequencies of normal modes of (a) longitudinal, and (b) transverse oscillations.

3 A double pendulum consisting of a pair, each of mass m and length l, is released from rest with the pendulums displaced but in a straight line. Find the displacements of the pendulums as functions of time.

4 Each of the pendulums in Fig. 12.2 is subjected to a damping force, of magnitude $\alpha\dot{x}$ and $\alpha\dot{y}$ respectively, while there is a damping force $\beta(\dot{x}-\dot{y})$ in the spring. Show that the equations for the normal co-ordinates q_1 and q_2 are still uncoupled. Find the amplitudes of the forced oscillations produced by applying a periodic force to one pendulum. Write down the general solution to the equations of motion, including transient terms. If the forcing frequency is that of the uncoupled pendulums, and the damping

in the spring is negligible, find the range of values of α for which the forced amplitude of the second pendulum is less than half that of the first.

5 If the damping constants α for the two pendulums in the preceding question are different, show that the equations of motion are no longer uncoupled. By substituting solutions of the form $q_\alpha = A_\alpha e^{pt}$, find an equation determining the (complex) values of p.

6 Show that a stretched string is equivalent mathematically to an infinite number of uncoupled oscillators, described by the co-ordinates

$$q_n = \left(\frac{2}{l}\right)^{1/2} \int_0^l y(x, t) \sin \frac{n\pi x}{l}\, dx.$$

Determine the amplitudes of the various normal modes in the motion described in Chapter 11, Problem 9. Why, physically, are the modes for even values of n not excited?

7 Show that a typical equation of the set (12.30) may be satisfied by setting $A_\alpha = \sin \alpha k$ ($\alpha = 1, \ldots, n$), provided that $\omega = 2\omega_0 \sin \frac{1}{2}k$. Hence show that the frequencies of the normal modes are $\omega_r = 2\omega_0 \sin [r\pi/2(n+1)]$, with $r = 1, 2, \ldots, n$. Why may we ignore values of r greater than $n+1$? Show that in the limit of large n, the frequency of the rth normal mode tends to the corresponding frequency for a continuous string with the same total length and mass.

8 A particle moves under a conservative force with potential energy $V(\mathbf{r})$. The point $\mathbf{r} = \mathbf{0}$ is a position of equilibrium, and the axes are chosen so that x, y, z are normal co-ordinates. Show that if V satisfies Laplace's equation, $\nabla^2 V = 0$, then the equilibrium is necessarily unstable, and hence that stable equilibrium under purely gravitational and electrostatic forces is impossible. (Of course, *dynamic* equilibrium—stable periodic motion—can occur.)

Hamiltonian Mechanics

Chapter 13

We have already seen, in several examples, the value of the Lagrangian method, which allows us to find equations of motion for any system in terms of an arbitrary set of generalized co-ordinates. In this chapter, we shall discuss an extension of the method, due to Hamilton. Its principal feature is the use of the generalized momenta $p_1, p_2, \ldots, p_n$, in place of the generalized velocities $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n$. It is particularly valuable when, as often happens, some of the generalized momenta are constants of the motion. More generally, it is well suited to finding conserved quantities, and making use of them.

13.1 Hamilton's Equations

The Lagrangian function L is a function of $q_1, q_2, \ldots, q_n$ and $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n$. For brevity, we shall indicate this dependence by writing $L(q, \dot{q})$, where q stands for all the generalized co-ordinates, and $\dot{q}$ for all their time derivatives.

Lagrange's equations may be written in the form

$$\dot{p}_\alpha = \frac{\partial L}{\partial q_\alpha}, \tag{13.1}$$

where the generalized momenta are defined by

$$p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha}. \tag{13.2}$$

Here, and in the following equations, α runs over $1, 2, \ldots, n$.

The instantaneous position and velocity of every part of our system may be specified by the value of the $2n$ variables q and $\dot{q}$. However, we can alternatively solve the equations (13.2) for the $\dot{q}$ in terms of q and p, obtaining, say

$$\dot{q}_\alpha = \dot{q}_\alpha(q, p), \tag{13.3}$$

and specify the instantaneous position and velocity by means of the $2n$ variables q and p.

For example, for a particle moving in a plane, and described by polar co-ordinates, $p_r = m\dot{r}$ and $p_\theta = mr^2\dot{\theta}$. In this case, the equations (13.3) read

$$\dot{r} = p_r/m, \qquad \dot{\theta} = p_\theta/mr^2. \tag{13.4}$$

The instantaneous position and velocity of the particle may be fixed by the values of r, θ, p_r and p_θ.

We now define a function of q and p, the *Hamiltonian function*, by

$$H = \sum_{\beta=1}^{n} p_\beta \dot{q}_\beta - L. \tag{13.5}$$

Here the variables $\dot{q}$ are to be regarded as functions of q and p, given by (13.3.) Written out to indicate the functional dependence, (13.5) reads

$$H(q, p) = \sum_{\beta=1}^{n} p_\beta \dot{q}_\beta(q, p) - L(q, \dot{q}(q, p)).$$

Next, we compute the derivatives of H. We differentiate first with respect to p_α. One term in this derivative is the coefficient of p_α in the sum $\Sigma p\dot{q}$, namely $\dot{q}_\alpha$. Other terms arise from the dependence of $\dot{q}$ on p_α. Altogether, we obtain

$$\frac{\partial H}{\partial p_\alpha} = \dot{q}_\alpha + \sum_{\beta=1}^{n} p_\beta \frac{\partial \dot{q}_\beta}{\partial p_\alpha} - \sum_{\beta=1}^{n} \frac{\partial L}{\partial \dot{q}_\beta}\frac{\partial \dot{q}_\beta}{\partial p_\alpha}.$$

Now, by (13.2), the second and third terms cancel. Hence we have

$$\frac{\partial H}{\partial p_\alpha} = \dot{q}_\alpha. \tag{13.6}$$

We now examine the derivative with respect to q_α. Again, there are two kinds of terms, the term coming from the explicit dependence of L on q_α, and those from the dependence of $\dot{q}$ on q_α. Thus

$$\frac{\partial H}{\partial q_\alpha} = -\frac{\partial L}{\partial q_\alpha} + \sum_{\beta=1}^{n} p_\beta \frac{\partial \dot{q}_\beta}{\partial q_\alpha} - \sum_{\beta=1}^{n} \frac{\partial L}{\partial \dot{q}_\beta}\frac{\partial \dot{q}_\beta}{\partial q_\alpha}.$$

As before, the second and third terms cancel. Using Lagrange's equations (13.1), we obtain

$$\frac{\partial H}{\partial q_\alpha} = -\dot{p}_\alpha. \tag{13.7}$$

The equations (13.6) and (13.7) together constitute *Hamilton's equations*. Whereas Lagrange's equations are a set of n second-order differential equations, these are a set of $2n$ first-order equations.

Let us consider, for example, a particle moving in a plane under a central conservative force. Then

$$L = \tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}mr^2\dot{\theta}^2 - V(r).$$

Hence the Hamiltonian function is

$$H = p_r\dot{r} + p_\theta\dot{\theta} - (\tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}mr^2\dot{\theta}^2 - V(r)),$$

or, using (13.4),

$$H = \frac{p_r^{\,2}}{2m} + \frac{p_\theta^{\,2}}{2mr^2} + V(r). \tag{13.8}$$

It may be noticed that this is the expression for the total energy, $T+V$. This is no accident, but a general property of natural systems, as we shall see below.

The first pair of Hamilton's equations are

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}, \qquad \dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2}. \tag{13.9}$$

They simply reproduce the relations (13.4) between velocities and momenta. The second pair are

$$-\dot{p}_r = \frac{\partial H}{\partial r} = -\frac{p_\theta^{\,2}}{mr^3} + \frac{\mathrm{d}V}{\mathrm{d}r}, \qquad -\dot{p}_\theta = \frac{\partial H}{\partial \theta} = 0. \tag{13.10}$$

The second of these two equations yields the law of conservation of angular momentum,

$$p_\theta = J = \text{constant}. \tag{13.11}$$

The first gives the radial equation of motion

$$\dot{p}_r = m\ddot{r} = \frac{J^2}{mr^3} - \frac{\mathrm{d}V}{\mathrm{d}r}.$$

It may be integrated to give the 'radial energy equation' of Chapter 4. (4.13) $\frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + V(r) = E.$

13.2 Conservation of Energy

We saw in §11.1 that a natural system is characterized by the fact that the kinetic energy contains no explicit dependence on the time, and is a homogeneous quadratic function of the time derivatives $\dot{q}$. page 164.
This condition may be expressed algebraically by the equation

$$\sum_{\alpha=1}^{n} \frac{\partial T}{\partial \dot{q}_\alpha} \dot{q}_\alpha = 2T.$$

For example, for $n = 2$, T has the form (12.1). Thus,

$$\frac{\partial T}{\partial \dot{q}_1}\dot{q}_1 + \frac{\partial T}{\partial \dot{q}_2}\dot{q}_2 = (a_{11}\dot{q}_1 + a_{12}\dot{q}_2)\dot{q}_1 + (a_{12}\dot{q}_1 + a_{22}\dot{q}_2)\dot{q}_2 = 2T.$$

Since $p_\alpha = \partial T/\partial \dot{q}_\alpha$, we therefore have

$$H = \sum_{\beta=1}^{n} p_\beta \dot{q}_\beta - L = \sum_{\beta=1}^{n} \frac{\partial T}{\partial \dot{q}_\beta}\dot{q}_\beta - (T-V)$$

$$= 2T - (T-V) = T+V.$$

p170 Pendulum Constrained to Rotate about an Axis. Paragraph 5 and equations (11.19b). Then equation (11.20) in paragraph 6.

Thus, for a natural system, the value of the Hamiltonian function is equal to the total energy of the system.

For a forced system, the Lagrangian can sometimes be written in the form $L = T' - V'$, where T' is a homogeneous quadratic in the variables $\dot{q}$, and V' is independent of them. (An example is the system discussed in §11.4.) In such a case, the Hamiltonian function H is equal to $T' + V'$, which is in general not the total energy.

Now let us evaluate the time derivative of H. We shall allow for the possibility that H may contain an explicit time dependence (as it does for some forced systems), and write $H = H(q, p, t)$. Then H varies with time for two reasons: firstly, because of its explicit dependence on t, and, secondly, because the variables q and p are themselves functions of time. Thus the total time derivative of H is

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial t} + \sum_{\alpha=1}^{n} \frac{\partial H}{\partial q_\alpha} \dot{q}_\alpha + \sum_{\alpha=1}^{n} \frac{\partial H}{\partial p_\alpha} \dot{p}_\alpha.$$

Now, if we express $\dot{q}$ and $\dot{p}$ in terms of derivatives of H, using Hamilton's equations (13.6) and (13.7), we obtain

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial t} + \sum_{\alpha=1}^{n} \left[\frac{\partial H}{\partial q_\alpha} \frac{\partial H}{\partial p_\alpha} - \frac{\partial H}{\partial p_\alpha} \frac{\partial H}{\partial q_\alpha} \right],$$

whence

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial t}. \tag{13.12}$$

This equation asserts that H changes with time *only* in virtue of its explicit time dependence. The net change induced by the fact that q and p vary with time is zero.

In particular, for a natural, conservative system, neither T nor V contains any explicit dependence on the time. Thus $\partial H/\partial t = 0$. It follows that

$$\frac{\mathrm{d}H}{\mathrm{d}t} = 0 \tag{13.13}$$

also, so that there is a law of conservation of energy,

$$H = T + V = E = \text{constant}. \tag{13.14}$$

(For some forced systems, too, it may happen that the Hamiltonian does not depend explicitly on t. In that case, we again have a conservation law—like (11.20)—though not for $T + V$.)

The Hamiltonian formalism is particularly well suited to finding conservation laws, or constants of the motion. The conservation law for energy is the first of a large class of conservation laws which we shall discuss in the following sections.

13.3 Ignorable Co-ordinates

It sometimes happens that one of the generalized co-ordinates, say q_α, does not appear in the Hamiltonian function. In that case, the co-ordinate q_α is said to be *ignorable*—for a reason we shall explain in a moment. For an ignorable co-ordinate, Hamilton's equation

$$-\dot{p}_\alpha = \frac{\partial H}{\partial q_\alpha} = 0 \tag{13.15}$$

leads immediately to a conservation law for the corresponding generalized momentum,

$$p_\alpha = \text{constant.} \tag{13.16}$$

For example, for a particle moving in a plane under a central conservative force, H is independent of the angular co-ordinate θ, and we therefore have the law of conservation of angular momentum (13.11).

The term 'ignorable co-ordinate' means just what is says—that for many purposes we can ignore the co-ordinate q_α, and treat the corresponding p_α simply as a constant appearing in the Hamiltonian function. This is, in effect, what we did for the central force problem in Chapter 4. Because of the conservation law for angular momentum, we were able to deal with an effectively one-dimensional problem involving only the radial co-ordinate r. The generalized momentum $p_\theta = J$ was simply a constant appearing in the equation of motion or the energy conservation equation.

Let us re-examine this problem from the Hamiltonian point of view. Since the Hamiltonian (13.8) is independent of θ, θ is ignorable. Thus we may regard (13.8) as the Hamiltonian for a system of *one* degree of freedom, described by the co-ordinate r, in which a constant p_θ appears. It is identical with the Hamiltonian for a particle moving in one dimension under a conservative force with potential energy function

$$U(r) = \frac{p_\theta^{\,2}}{2mr^2} + V(r). \tag{13.17}$$

This is the 'effective potential energy function' of (4.14).

Hamilton's equations for r and p_r are

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}, \qquad -\dot{p}_r = \frac{\partial H}{\partial r} = \frac{\mathrm{d}U}{\mathrm{d}r}.$$

To solve the central force problem, we solve first this one-dimensional problem (for example by using the energy conservation equation for

this problem). Our solution gives us complete information about the radial motion—it gives $\dot{r}$ as a function of r, and therefore r as a function of t, by integrating.

Any required information about the angular part of the motion can then be found from the remaining pair of Hamilton's equations, one of which is the angular momentum conservation equation $\dot{p}_\theta = 0$, while the other gives $\dot{\theta}$ in terms of p_θ, $\dot{\theta} = p_\theta/mr^2$. Clearly, though we did not introduce the Hamiltonian, this is essentially just the method we used in Chapter 4.

We shall use the same method in the following section to discuss the general motion of a symmetric top.

13.4 The Symmetric Top

We start from the Lagrangian function (11.11) for this system. The corresponding generalized momenta are

$$\begin{aligned} p_\varphi &= I_1\dot{\varphi}\sin^2\theta + I_3(\dot{\psi}+\dot{\varphi}\cos\theta)\cos\theta, \\ p_\theta &= I_1\dot{\theta}, \\ p_\psi &= I_3(\dot{\psi}+\dot{\varphi}\cos\theta). \end{aligned}$$

Solving these equations for $\dot{\varphi}$, $\dot{\theta}$, $\dot{\psi}$, we obtain

$$\begin{aligned} \dot{\varphi} &= \frac{p_\varphi - p_\psi\cos\theta}{I_1\sin^2\theta}, \\ \dot{\theta} &= \frac{p_\theta}{I_1}, \\ \dot{\psi} &= \frac{p_\psi}{I_3} - \frac{p_\varphi - p_\psi\cos\theta}{I_1\sin^2\theta}\cos\theta. \end{aligned} \tag{13.18}$$

The simplest way to obtain the Hamiltonian function is to use the fact that $H = T+V$, and express T in terms of the generalized momenta by using (13.18). In this way, we find

$$H = \frac{(p_\varphi - p_\psi\cos\theta)^2}{2I_1\sin^2\theta} + \frac{p_\theta^{\,2}}{2I_1} + \frac{p_\psi^{\,2}}{2I_3} + Mgr\cos\theta. \tag{13.19}$$

It is easy to verify that the first set of Hamilton's equations, (13.6), correctly reproduce (13.18).

Two of the three co-ordinates here are ignorable, and there are two corresponding conservation laws $p_\varphi = \text{constant}$, and $p_\psi = \text{constant}$. Thus the problem can be reduced to that of a system with

one degree of freedom only, described by the co-ordinate θ. The Hamiltonian (13.19) may be written

$$H = \frac{p_\theta^2}{2I_1} + U(\theta), \tag{13.20}$$

where the effective potential energy function $U(\theta)$ is

$$U(\theta) = \frac{(p_\varphi - p_\psi \cos\theta)^2}{2I_1 \sin^2\theta} + \frac{p_\psi^2}{2I_3} + Mgr\cos\theta. \tag{13.21}$$

The last of Hamilton's equations is

$$-I_1\ddot{\theta} = -\dot{p}_\theta = \frac{\partial H}{\partial\theta} = \frac{dU}{d\theta}. \tag{13.22}$$

This is obviously a rather complicated equation to solve. However, the qualitative features of the motion can be found from the energy conservation equation

$$\frac{p_\theta^2}{2I_1} + U(\theta) = E = \text{constant}. \tag{13.23}$$

The angles θ for which $\dot{\theta} = 0$ are given by the equation $U(\theta) = E$, and the motion is confined to the region where $U(\theta) \leqslant E$.

Now let us examine the function $U(\theta)$. We exclude for the moment the case where $p_\varphi = \pm p_\psi$. (We return to this special case in the next section.) Then it is clear from (13.21) that as θ approaches 0 or π, $U(\theta) \to +\infty$. Hence it has roughly the form shown in Fig. 13.1, with a minimum at some value of θ, say θ_0, between 0 and π.* When E is equal to this minimum value, we have an 'equilibrium' situation, and θ remains fixed at θ_0. This corresponds to steady precession. For any larger value of E, the angle θ oscillates between a minimum θ_1 and a maximum θ_2.

It is not hard to describe the motion of the top. We note that, according to (13.18), the angular velocity of the axis about the vertical $\dot{\varphi}$, is zero when $\cos\theta = p_\varphi/p_\psi$. If this angle lies outside the range between θ_1 and θ_2, then $\dot{\varphi}$ never vanishes, and the axis precesses round the vertical in a fixed direction, and wobbles up and down between θ_1 and θ_2. This motion is illustrated in Fig. 13.2 which shows the position of the end of the axis on a sphere.

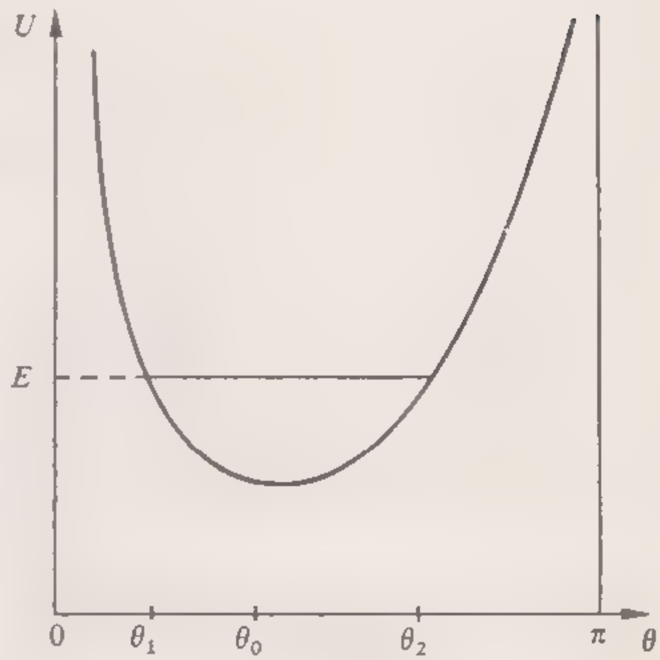


Fig. 13.1

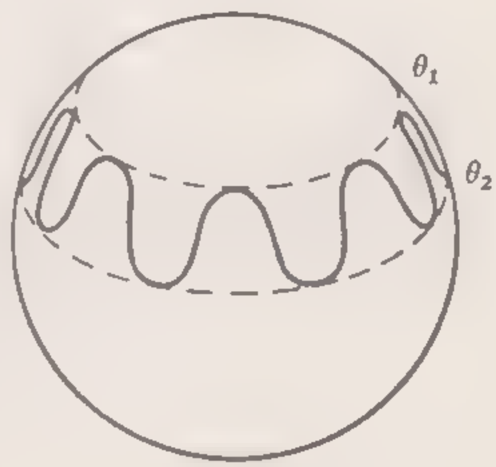


Fig. 13.2

* One can show that there is only *one* minimum as follows. The equation $U(\theta) = E$ may be written, in terms of the variable $z = \cos\theta$, as $(p_\varphi - p_\psi z)^2 - 2I_1(1-z^2)(E - Mgrz - p_\psi^2/2I_3) = 0$. This is a cubic equation for z, with three roots in general. However, the left side is positive at both $z = 1$ and $z = -1$. Thus there are either two roots or none between these points. It follows that for every value of E, there are at most two angles θ for which $U(\theta) = E$.

On the other hand, if $\theta_1 < \cos^{-1}(p_\varphi/p_\psi) < \theta_2$, the axis moves in loops, as shown in Fig. 13.3. The angular velocity $\dot{\phi}$ has one sign near the top of the loop, and the opposite sign near the bottom.

The limiting case between the two kinds of motion occurs when $\cos^{-1}(p_\varphi\, p_\psi) = \theta_1$. Then the loops shrink to cusps, as shown in Fig. 13.4. The axis of the top comes instantaneously to rest at the top of each loop. This kind of motion will occur if the top is set spinning with its axis initially at rest. (It is impossible to have cusped motion with the cusps at the bottom, for they correspond to points of minimum kinetic energy, and the motion must always be below such points. A top set spinning with its axis stationary cannot rise without increasing its energy.)*

It is easy to observe these kinds of motion using a small gyroscope. In practice, because of frictional effects, the type of motion will change slowly with time.

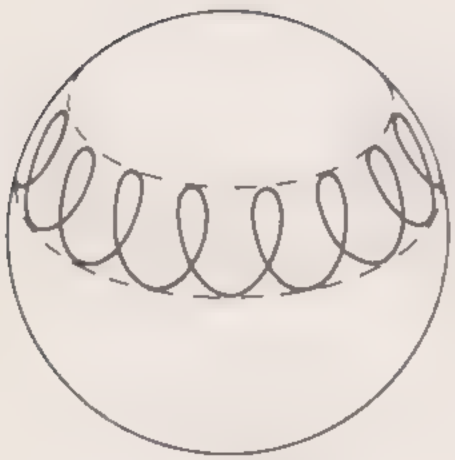


Fig. 13.3

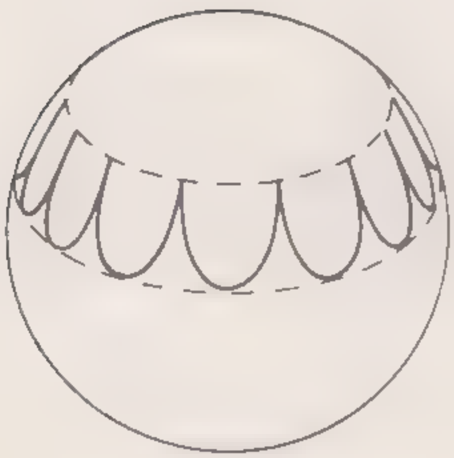


Fig. 13.4

13.5 Stability of a Vertical Top

If the axis of the top passes through the vertical, $\theta = 0$, then it is clear that $U(0)$ must be finite. This is possible only if $p_\varphi = p_\psi$, and both generalized momenta must be equal to $I_3\omega_3$. (We could also consider types of motion for which the axis passes through the downward vertical. In that case, we require $p_\varphi = -p_\psi = -I_3\omega_3$. The treatment is entirely similar.)

If we set $p_\varphi = p_\psi = I_3\omega_3$, the effective potential energy function (13.21) becomes

$$U(\theta) = \frac{I_3^{\,2}\omega_3^{\,2}}{2I_1} \tan^2 \tfrac{1}{2}\theta + \tfrac{1}{2}I_3\omega_3^{\,2} + Mgr\cos\theta, \qquad (13.24)$$

where we have used the identity $(1-\cos\theta)/\sin\theta = \tan\tfrac{1}{2}\theta$.

For small values of θ, we may expand $U(\theta)$, and retain only the terms up to order θ^2. We obtain

$$U(\theta) = (\tfrac{1}{2}I_3\omega_3^{\,2} + Mgr) + \frac{1}{2}\left(\frac{I_3^{\,2}\omega_3^{\,2}}{4I_1} - Mgr\right)\theta^2.$$

Since there is no linear term, $\theta = 0$ is always a position of equilibrium. It is a position of *stable* equilibrium if $U(\theta)$ has a minimum at $\theta = 0$, that is, if the coefficient of θ^2 is positive. Thus there is a minimum value of ω_3 for which the vertical top is stable, given by

$$\omega_3^{\,2} = \frac{4I_1 Mgr}{I_3^{\,2}} = \omega_0^{\,2}, \quad \text{say.} \qquad (13.25)$$

* See, however, Chapter 10, Problem 5 (p. 162).

If the top is set spinning with its axis vertical, then the axis will remain vertical so long as $\omega_3 > \omega_0$, but when the angular velocity falls below this value (as it eventually will, because of friction), the top will begin to wobble. The energy of the vertical top is

$$E = \tfrac{1}{2} I_3 \omega_3^2 + Mgr.$$

Thus the angles for which $\dot{\theta} = 0$ are given by $U(\theta) = E$ (see Fig. 13.5), or, from (13.24), by

$$\frac{I_3^2 \omega_3^2}{2I_1} \tan^2 \tfrac{1}{2}\theta = Mgr(1 - \cos\theta) = 2Mgr \sin^2 \tfrac{1}{2}\theta.$$

They are $\theta_1 = 0$, and $\theta_2 = 2 \cos^{-1}(\omega_3/\omega_0)$. Thus, if the top is set spinning with its axis vertical and almost stationary, with angular velocity less than the critical value ω_0, it will oscillate in the subsequent motion between the vertical and the angle θ_2. Note that θ_2 increases as ω_3 is decreased, and tends to π as ω_3 approaches zero. When $\omega_3 = 0$, the top behaves like a compound pendulum, and swings in a circle through both the upward and downward verticals.

13.6 Symmetries and Conservation Laws

In §§13.2 and 13.3, we found some examples of conserved quantities, but so far we have not discussed the physical reasons for their existence. In fact, they are expressions of symmetry properties possessed by the system. (13-14), (13-16)

For example, the conservation law of angular momentum for the central force problem arises from the fact that the Hamiltonian is independent of θ. This is an expression of the rotational symmetry of the system—or, in other words, of the fact that there is no preferred orientation in the plane. Explicitly, the equation $\partial H \, \partial\theta = 0$ means that the energy of the system is unchanged if we rotate it to a new position, replacing θ by $\theta + \delta\theta$, without changing r, p_r or p_θ. Thus angular momentum is conserved (p_θ = constant) for systems possessing this rotational symmetry. Of course, if the force is non-central, it does determine a preferred orientation in space, and angular momentum is not conserved.

In the three-dimensional problem, every component of the angular momentum $\mathbf{J}$ is conserved if the force is purely central. If the force is non-central, but still possesses axial symmetry—so that H depends on θ but not φ—then only the component of $\mathbf{J}$ along the axis of symmetry, namely p_φ, is conserved.

Similarly, for the symmetric top, the equation $\partial H/\partial\varphi = 0$ is an expression of the rotational symmetry of the system about the verti-

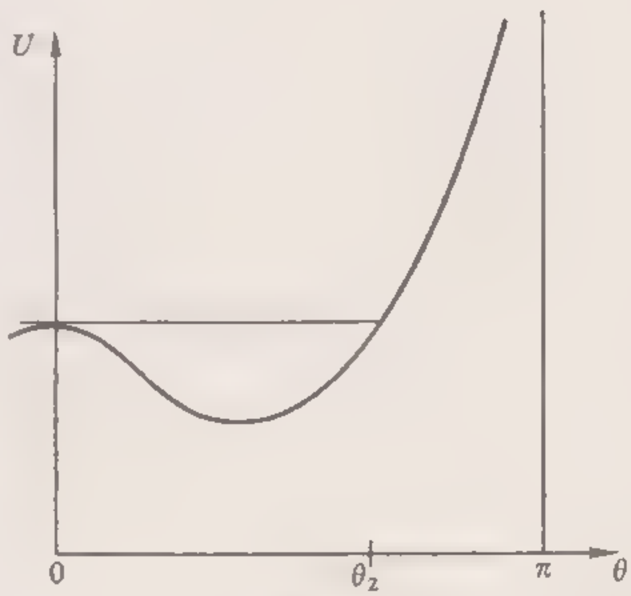


Fig. 13.5

cal. The corresponding conserved quantity p_φ is the vertical component of **J**; for, by (10.20) and (10.22),

$$J_z = \mathbf{k}\cdot\mathbf{J} = I_1\dot\phi \sin^2\theta + I_3(\dot\psi+\dot\phi\cos\theta)\cos\theta = p_\varphi.$$

The equation $\partial H/\partial\psi = 0$ expresses the rotational symmetry of the top itself about its own axis. The energy is unchanged by rotating the top about this axis. In this case, we see from (10.22) that the conserved quantity p_φ is the component of **J** along the axis of the top, $J_3 = \mathbf{e}_3\cdot\mathbf{J}$.

Now of course not all symmetry properties are expressible simply by saying that H is independent of some particular co-ordinate. For example, we might consider the central force problem in terms of Cartesian co-ordinates x and y. Then the Hamiltonian is

$$H = \frac{p_x{}^2+p_y{}^2}{2m}+V[(x^2+y^2)^{1/2}].$$

Since it depends on both x and y, neither co-ordinate is ignorable. It does, however, possess a symmetry property under rotations. If we make a small rotation through an angle $\delta\theta$, the changes in the co-ordinates and momenta are

$$\begin{aligned} \delta x &= -y\delta\theta, & \delta y &= x\delta\theta, \\ \delta p_x &= -p_y\delta\theta, & \delta p_y &= p_x\delta\theta. \end{aligned} \tag{13.26}$$

Under this transformation, $\delta(x^2+y^2) = 0$, and $\delta(p_x{}^2+p_y{}^2) = 0$, so that clearly $\delta H = 0$.

Now we know from our earlier discussion in terms of polar co-ordinates that this symmetry is related to the conservation of angular momentum,

(5-27)

$$J = xp_y - yp_x = \text{constant}.$$

The problem is to understand the relationship between the transformation (13.26) and the conserved quantity J.

Let us consider a general function of the co-ordinates, momenta and time, $G(q, p, t)$. We define the transformation *generated* by G to be

$$\delta q_\alpha = \frac{\partial G}{\partial p_\alpha}\,\delta\lambda, \qquad \delta p_\alpha = -\frac{\partial G}{\partial q_\alpha}\,\delta\lambda, \tag{13.27}$$

where $\delta\lambda$ is an infinitesimal parameter. For example the function $G = p_1$ generates the transformation in which $\delta q_1 = \delta\lambda$, while all the remaining co-ordinates and momenta are unchanged. The Hamiltonian H generates the transformation

$$\delta q_\alpha = \dot q_\alpha\,\delta\lambda, \qquad \delta p_\alpha = \dot p_\alpha\,\delta\lambda. \tag{13.28}$$

If $\delta\lambda$ is interpreted as a small time interval, this represents the time development of the system.

We can now return to the function J. The transformation it generates is given by

$$\delta x = \frac{\partial J}{\partial p_x}\,\delta\lambda = -y\delta\lambda, \qquad \delta y = \frac{\partial J}{\partial p_y}\,\delta\lambda = x\delta\lambda,$$

$$\delta p_x = -\frac{\partial J}{\partial x}\,\delta\lambda = -p_y\delta\lambda, \qquad \delta p_y = -\frac{\partial J}{\partial y}\,\delta\lambda = p_x\delta\lambda.$$

This is clearly identical with the infinitesimal rotation (13.26). Thus we have established a connection between J and the transformation (13.26). ?

The next problem is to understand why the fact that this transformation represents a symmetry property of the system should lead to a conservation law. To this end, we return to a general function G, and consider the effect of the transformation (13.27) on some other function $F(q, p, t)$. The change in F is

$$\begin{aligned}\delta F &= \sum_{\alpha=1}^{n}\left(\frac{\partial F}{\partial q_\alpha}\,\delta q_\alpha + \frac{\partial F}{\partial p_\alpha}\,\delta p_\alpha\right)\\ &= \sum_{\alpha=1}^{n}\left(\frac{\partial F}{\partial q_\alpha}\frac{\partial G}{\partial p_\alpha} - \frac{\partial F}{\partial p_\alpha}\frac{\partial G}{\partial q_\alpha}\right)\delta\lambda.\end{aligned}$$

This kind of sum appears quite frequently, and it is therefore convenient to introduce an abbreviated notation. We define the *Poisson bracket* of F and G to be

$$[F, G] = \sum_{\alpha=1}^{n}\left(\frac{\partial F}{\partial q_\alpha}\frac{\partial G}{\partial p_\alpha} - \frac{\partial F}{\partial p_\alpha}\frac{\partial G}{\partial q_\alpha}\right). \tag{13.29}$$

a surely?

Then we can write the change in F under the transformation generated by G in the form

$$\delta F = [F, G]\,\delta\lambda. \tag{13.30}$$

A particular example is provided by the transformation (13.28) generated by H. The rate of change of F is

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \sum_{\alpha=1}^{n}\left(\frac{\partial F}{\partial q_\alpha}\,\dot{q}_\alpha + \frac{\partial F}{\partial p_\alpha}\,\dot{p}_\alpha\right) = \frac{\partial F}{\partial t} + [F, H]. \tag{13.31}$$

The extra term here arises from the fact that we have allowed F to have an explicit dependence on the parameter t, in addition to the dependence via q and p. ?

Now an obvious property of the Poisson bracket (13.29) is its *antisymmetry*. If we interchange F and G, we merely change the sign:

$$[G, F] = -[F, G]. \tag{13.32}$$

This has the important consequence that, if F is unchanged by the transformation generated by G, then reciprocally G is unchanged by the transformation generated by F.

We are now finally in a position to apply this discussion to the case of a symmetry property of the system. Let us suppose that there exists a transformation of the co-ordinates and momenta which leaves the Hamiltonian unaffected, and which is generated by a function G. From (13.27) we see that the generator G is unique, apart from an arbitrary additive function of t, independent of q or p. In particular, if the transformation does not involve t explicitly, then G may be chosen to contain no explicit t dependence. The condition that H should be unchanged is

$$\delta H = [H, G]\,\delta\lambda = 0. \tag{13.33}$$

It then follows from the reciprocity relation (13.32) that $[G, H] = 0$ also. Hence, if $\partial G/\partial t = 0$, we find from (13.31) that

$$\frac{\mathrm{d}G}{\mathrm{d}t} = 0. \tag{13.34}$$

Thus we have shown that if H is unaltered by a transformation of this type, then the corresponding generator is conserved.

13.7 Galilean Transformations

To illustrate the ideas of the preceding section, we shall consider a general system of N particles, and investigate the symmetry properties implied by the relativity principle of §1.1.

The system has $3N$ degrees of freedom, and may be described by the particle positions $\mathbf{r}_i$ and momenta $\mathbf{p}_i$. We shall consider four distinct symmetry properties, associated with the requirements that there should be no preferred zero of the time scale; origin in space, orientation of axes, or standard of rest. The corresponding symmetry transformations are translations in time, spatial translations, rotations, and transformations between frames moving with uniform relative velocity. A combination of these four types of transformation is the most general transformation which takes one inertial frame into another. They are known collectively as *Galilean* transformations. We shall consider them in turn.

Time Translations. The changes in **r** and **p** in an infinitesimal time δt are generated by the Hamiltonian function H. The condition for invariance of H under this transformation is $[H, H] = 0$, which is certainly true because of (13.32). Thus, as we showed in §13.2, if H contains no explicit time dependence, then it is in fact conserved,

$$\frac{\mathrm{d}H}{\mathrm{d}t} = 0. \tag{13.35}$$

and (13.13)

Spatial Translations. An infinitesimal translation of the system through a distance δx in the x direction is represented by the transformation

$$\begin{aligned} \delta x_i &= \delta x, & \delta y_i &= 0, & \delta z_i &= 0, \\ \delta p_{xi} &= 0, & \partial p_{yi} &= 0, & \delta p_{zi} &= 0. \end{aligned} \tag{13.36}$$

The corresponding generator is easily seen to be

$$P_x = \sum_{i=1}^{N} p_{xi}.$$

The condition for H to be invariant under this transformation is

$$0 = [H, P_x]\,\delta x = \sum_{i=1}^{N} \frac{\partial H}{\partial x_i}\,\delta x.$$

It is satisfied if H depends only on the co-ordinate differences $x_i - x_j$; for then changing each x_i by the same amount does not affect H. When this condition holds, we obtain a conservation law for the x component of momentum, $\mathrm{d}P_x/\mathrm{d}t = 0$.

More generally, a translation in the direction of the unit vector **n** is generated by the component of the total momentum **P** in this direction, **n**·**P**. When the system possesses translational invariance in all directions, then all components of **P** are conserved:

$$\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t} = \mathbf{0}. \tag{13.37}$$

This is physically very reasonable. We know from our earlier work that momentum is conserved for an isolated system, but not for a system subjected to external forces which determine a preferred origin (for example, a centre of force.)

Rotations. An infinitesimal rotation through an angle $\delta\varphi$ about the z-axis yields

$$\begin{aligned} \delta x_i &= -y_i\,\delta\varphi, & \delta y_i &= x_i\,\delta\varphi, & \delta z_i &= 0, \\ \delta p_{xi} &= -p_{yi}\,\delta\varphi, & \delta p_{yi} &= p_{xi}\,\delta\varphi, & \delta p_{zi} &= 0. \end{aligned} \tag{13.38}$$

The corresponding generator is the z component of the total angular momentum,

$$J_z = \sum_{i=1}^{N} (x_i p_{yi} - y_i p_{xi}).$$

The condition for H to be rotationally symmetric, $[H, J_z] = 0$, is satisfied provided that H involves the x and y co-ordinates and momentum components only through invariant combinations like $x_i x_j + y_i y_j$.

In general, a rotation through φ about an axis in the direction of the unit vector $\mathbf{n}$ may be written in the form

$$\delta\mathbf{r}_i = \mathbf{n} \wedge \mathbf{r}_i \, \delta\varphi, \qquad \delta\mathbf{p}_i = \mathbf{n} \wedge \mathbf{p}_i \, \delta\varphi. \tag{13.39}$$

It is generated by the appropriate component of angular momentum, $\mathbf{n} \cdot \mathbf{J}$. The Hamiltonian is invariant under this transformation provided that it is a *scalar* function of $\mathbf{r}$ and $\mathbf{p}$, involving only the squares and scalar products of vectors. It is easy to verify that scalar quantities are indeed unchanged by (13.39). For example,

$$\begin{aligned}\delta(\mathbf{r}_i \cdot \mathbf{r}_j) &= (\delta\mathbf{r}_i) \cdot \mathbf{r}_j + \mathbf{r}_i \cdot (\delta\mathbf{r}_j) \\ &= [(\mathbf{n} \wedge \mathbf{r}_i) \cdot \mathbf{r}_j + \mathbf{r}_i \cdot (\mathbf{n} \wedge \mathbf{r}_j)] \, \delta\varphi = 0.\end{aligned}$$

When the Hamiltonian possesses complete rotational symmetry, then each component of $\mathbf{J}$ is conserved, and so

$$\frac{\mathrm{d}\mathbf{J}}{\mathrm{d}t} = \mathbf{0}. \tag{13.40}$$

So far, we have shown that the conservation laws of energy, momentum and angular momentum are expressions of symmetry properties required by the relativity principle. They have, therefore, a much more general validity than the specific assumptions used in their original derivation. For example, we have not assumed that the forces in our system are all two-body forces, or that they are central or conservative. All we have assumed is the existence of the Hamiltonian function, and the relativity principle.

There remains one type of Galilean transformation, which is in some respects rather different from the others.

Transformations to Moving Frames. Let us consider the effect of giving our system a small overall velocity δv in the x direction. The corresponding transformations are

$$\begin{aligned} &\delta x_i = t\delta v, \qquad \delta y_i = 0, \qquad \delta z_i = 0, \\ &\delta p_{xi} = m_i \, \delta v, \quad \delta p_{yi} = 0, \quad \delta p_{zi} = 0. \end{aligned} \tag{13.41}$$

This transformation differs from the others we have considered in that t appears explicitly in (13.41). The generator of this transformation is

$$G_x = \sum_{i=1}^{N} (p_{xi}t - m_i x_i) = P_x t - MX,$$

where X is the x co-ordinate of the centre of mass. This generator is also explicitly time-dependent. It is clearly the x component of the vector

$$\mathbf{G} = \mathbf{P}t - M\mathbf{R}. \tag{13.42}$$

Transformations in other directions are generated by appropriate components of this vector.

We must now be careful. For, because of the explicit time-dependence, it is no longer true that H must be invariant under (13.41). Indeed, we know that the energy of a system does depend on the choice of reference frame, though not on the choice of origin or axes. What the relativity principle actually requires is that the *equations of motion* should be unchanged by the transformation. It can be shown that the necessary condition for this is still*

$$\frac{\mathrm{d}G}{\mathrm{d}t} = 0. \tag{13.43}$$

* The proof, in outline, is as follows. The first set of Hamilton's equations, (13.6), will be unchanged provided that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\delta q_\alpha) = \delta\left(\frac{\partial H}{\partial p_\alpha}\right),$$

or, in terms of the generator G,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial G}{\partial p_\alpha} = \left[\frac{\partial H}{\partial p_\alpha}, G\right] = -\left[G, \frac{\partial H}{\partial p_\alpha}\right],$$

by (13.32). Thus, using (13.31) for the left side,

$$0 = \frac{\partial^2 G}{\partial t\, \partial p_\alpha} + \left[\frac{\partial G}{\partial p_\alpha}, H\right] + \left[G, \frac{\partial H}{\partial p_\alpha}\right]$$

$$= \frac{\partial}{\partial p_\alpha}\left(\frac{\partial G}{\partial t} + [G, H]\right) = \frac{\partial}{\partial p_\alpha}\left(\frac{\mathrm{d}G}{\mathrm{d}t}\right)$$

Thus $\mathrm{d}G/\mathrm{d}t$ is independent of each p_α. Similarly, by using the other set of Hamilton's equations, (13.7), one can show that it is independent of all the q_α. Since we can always add to G a function of t alone without affecting the transformations it generates, we can choose it so that $\mathrm{d}G/\mathrm{d}t = 0$.

However, this no longer implies the invariance of H. In fact, using (13.31) and (13.32), we find

$$\delta H = [H, G]\,\delta\lambda = -[G, H]\,\delta\lambda = \frac{\partial G}{\partial t}\,\delta\lambda. \tag{13.44}$$

Thus the change in H is related to the explicit time-dependence of G.

The fourth of the basic conservation laws, for the quantity (13.42), is

$$\frac{\mathrm{d}\mathbf{G}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{P}t - M\mathbf{R}) = \mathbf{0}. \tag{13.45}$$

Though this is an unfamiliar form, the equation is actually well known. For, since $\mathrm{d}\mathbf{P}/\mathrm{d}t = \mathbf{0}$, it may be written

$$\mathbf{P} - M\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} = \mathbf{0},$$

which is simply the relation between total momentum and centre-of-mass velocity. This relation too is therefore a consequence of the relativity principle.

For a transformation in the x direction, the change in H, given by (13.44), is

$$\delta H = \frac{\partial G_x}{\partial t}\,\delta v = P_x\,\delta v. \tag{13.46}$$

If we write $H = T + V$, where

$$T = \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2m_i},$$

then we find directly from (13.41) that

$$\delta T = \sum_{i=1}^{N} p_{xi}\,\delta v = P_x\,\delta v.$$

Thus the change in the kinetic energy is exactly what is demanded by (13.46), and this relation reduces to

$$\delta V = 0. \tag{13.47}$$

It is interesting to examine the conditions imposed on V by this requirement. We have already seen that V must be a scalar function, and must involve the particle positions only through the differences $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$. Since these differences are unaffected by (13.41), the requirement (13.47) imposes no further restrictions on the $\mathbf{r}$ dependence. Moreover, V must not contain any explicit dependence on time. However, none of the conditions imposed so far requires it to

be independent of the momenta. To satisfy (13.47) it must contain them only through the invariant combinations

$$\frac{\mathbf{p}_i}{m_i}-\frac{\mathbf{p}_j}{m_j},$$

which are easily seen to be invariant under (13.41), but this is the only new requirement. Thus the most general form of interaction in an N-particle system which is invariant under all Galilean transformations is one described by a 'potential energy function' V which is an arbitrary scalar function of the relative position vectors *and* the relative velocity vectors.

Reflections; Parity. There is one other type of co-ordinate transformation which might be mentioned. All those we have considered so far have the property that if we start with a right-handed set of axes, then the transformation takes us to another right-handed set. However, we could also consider transformations like reflections (say $x \to -x$, $y \to y$, $z \to z$) or inversions ($\mathbf{r} \to -\mathbf{r}$) which lead from a right-handed set of axes to a left-handed set. These are called *improper* co-ordinate transformations. They differ from the proper transformations, such as rotations, in being discrete rather than continuous—no continuous change can ever take a right-handed set of axes into a left-handed one.

The condition for the Hamiltonian function to be unchanged under improper co-ordinate transformations is that it should be a true scalar function, like $\mathbf{r}_i \cdot \mathbf{r}_j$, rather than a *pseudo*scalar, like $(\mathbf{r}_i \wedge \mathbf{r}_j) \cdot \mathbf{r}_k$, which changes sign under inversion. If this condition is fulfilled, the equations of motion will have the same form in right-handed and left-handed frames of reference.

Because of the discontinuous nature of these transformations, this symmetry does not lead to a conservation law for some continuous variable. In fact, in classical mechanics, it does not lead to a conservation law at all. However, in quantum mechanics it yields a conservation law for a quantity known as the *parity*, which has only two possible values, ± 1. Until 1957, it was believed that all physical laws were unchanged by reflections, but it was then discovered that parity is in fact not conserved in the process of radioactive decay of atomic nuclei. The laws describing such processes do not have the same form in right-handed and left-handed frames of reference.

13.8 Summary

The Hamiltonian method is an extremely powerful tool in dealing with complex problems. In particular, when the Hamiltonian func-

tion is independent of some particular co-ordinate q_α, then the corresponding generalized momentum p_α is conserved. In such a case, the number of degrees of freedom is effectively reduced by one.

More generally, we have seen that any symmetry property of the system leads to a corresponding conservation law. This can be of great importance in practice, since the amount of labour involved in solving a complicated problem can be greatly reduced by making full use of all the available symmetries.

The Hamiltonian function is also of great importance in quantum mechanics, and many of the features of our discussion carry over to that case. We have seen that the variables appear in pairs. To each co-ordinate q_α there corresponds a momentum p_α. Such pairs are called *canonically conjugate*. This relationship between pairs of variables is of central importance in quantum mechanics, where there is an 'uncertainty principle' according to which it is impossible to measure both members of such a pair simultaneously with arbitrary accuracy.

The relationship between symmetries and conservation laws also applies to quantum mechanics. In relativity, the transformations we consider are slightly different (Lorentz transformations rather than Galilean), but the same principles apply, and lead to very similar conservation laws.

It is fitting that the book should end, as it began, with the relativity principle. The relationship between this principle and the familiar conservation laws (including the 'conservation law' $\mathbf{P} = M\dot{\mathbf{R}}$) is of the greatest importance for the whole of physics. It is the basic reason for the universal character of these laws, which were originally derived as rather special consequences of Newton's laws, but can now be seen as having a far more fundamental role.

PROBLEMS

1 A particle of mass m slides on the inside of a smooth cone of semi-vertical angle α, whose axis points vertically upward. Obtain the Hamiltonian function, using the distance r from the vertex, and the azimuth angle φ as generalized co-ordinates. Show that stable circular motion is possible for any value of r, and determine the corresponding angular velocity, ω. Find the angle α if the angular frequency of small oscillations about this circular motion is also ω.

2 Find the Hamiltonian function for the forced pendulum considered in §11.4, and verify that it is equal to $T'+V'$. Determine the frequency of small oscillations about the stable position when $\omega^2 > g/l$.

3 A light string passes over a small pulley and carries a mass $2m$ on one end. On the other is a mass m, and beneath it, supported by a spring with spring constant k, a second mass m. Find the Hamiltonian function, using

the distance x of the first mass m beneath the pulley, and the extension y in the spring, as generalized co-ordinates. Show that x is ignorable. To what symmetry property does this correspond? (In other words, what operation can be performed on the system without changing its energy?) If the system is released from rest with the spring unextended, find the positions of the particles at any later time.

4 A particle of mass m moves in three dimensions under the conservative force with potential energy $V(r)$. Find the Hamiltonian function in terms of spherical polar co-ordinates, and show that φ but not θ is ignorable. Express the quantity $\mathbf{J}^2 = m^2r^4[\dot{\theta}^2+\sin^2\theta\dot{\varphi}^2]$ in terms of the generalized momenta, and show that it is a second constant of the motion.

5 Obtain the Hamiltonian function for the top with freely sliding pivot described in Chapter 11, Problem 7. Find whether the minimum angular velocity required for stable vertical rotation is greater or less than in the case of a fixed pivot. Can you explain this result physically?

6 Show that the Hamiltonian function for a particle of charge q in an electromagnetic field is

$$H = \frac{1}{2m}\left(\mathbf{p}-\frac{q}{c}\mathbf{A}\right)^2+q\phi$$

By starting from the Lagrangian function in cylindrical polars, show that in an axially symmetric magnetic field, described by the single component $A_\varphi(\rho, z)$, it takes the form

$$H = \frac{1}{2m}\left[p_z^{\,2}+p_\rho^{\,2}+\frac{1}{\rho^2}\left(p_\varphi-\frac{q}{c}\rho A_\varphi\right)^2\right].$$

(Note that the subscripts φ on the generalized momentum p_φ and the component A_φ mean different things.)

7 A particle of mass m and charge q is moving in the equatorial plane $z=0$ of a magnetic dipole of moment μ. (See (B.17).) Show that it will continue to move in this plane. Initially, it is approaching from a great distance with velocity v and impact parameter b, whose sign is defined to be that of p_φ. Show that v and p_φ are constants of the motion, and that the distance of closest approach to the dipole is $\frac{1}{2}[(b^2\mp a^2)^{1/2}\pm b]$, according as $b>a$ or $b<a$, where $a^2 = 4q\mu/mcv$. (Here $q\mu$ is assumed positive.) Find also the range of values of b for which the velocity can become purely radial, and the distances at which it does so. Describe qualitatively the appearance of the orbits for different values of b.

8 The magnetic field in an accelerator is axially symmetric, and in the plane $z = 0$ has only a z component.

If $J = p_\varphi-(q/c)\rho A_\varphi$, show, using (B.6) and (A.47) that $\partial J/\partial\rho = -(q/c)\rho B_z$ and $\partial J/\partial z = (q/c)\rho B_\rho$. What is the relation between $\dot{\varphi}$ and J? Treat the third term of the Hamiltonian in Problem 6 as an effective potential energy $U(\rho, z) = J^2/2m\rho^2$, compute its derivatives, and write down the 'equilibrium' conditions $\partial U/\partial\rho = \partial U/\partial z = 0$. Hence show that a particle of mass m and charge q can move in a circle of any given radius a in the plane $z = 0$ with angular velocity equal to the cyclotron frequency for the field at that radius.

9 To investigate the stability of the motion described in the preceding question, evaluate the second derivatives of U at $\rho = a$, $z = 0$ and show

that they may be written

$$\frac{\partial^2 U}{\partial \rho^2} = \frac{1}{m}\left(\frac{q}{c}\right)^2 \left[B_z\left(B_z + \rho\frac{\partial B_z}{\partial \rho}\right)\right]_{\rho=a,\ z=0},$$

$$\frac{\partial^2 U}{\partial \rho\, \partial z} = 0, \quad \frac{\partial^2 U}{\partial z^2} = -\frac{1}{m}\left(\frac{q}{c}\right)^2 \left[B_z \rho \frac{\partial B_z}{\partial \rho}\right]_{\rho=a,\ z=0}$$

(You will need to use the φ component of the equation $\nabla \wedge \mathbf{B} = \mathbf{0}$, and the fact that, since $B_\rho = 0$ for all ρ, $\partial B_\rho/\partial \rho = 0$ also.) If the dependence of B_z on ρ near the equilibrium orbit is described by $B_z \propto (a/\rho)^n$, show that the orbit is stable only if $0 < n < 1$.

10 Show that the Poisson brackets of the components of angular momentum are

$$[J_x, J_y] = J_z.$$

Interpret this result in terms of the transformation of one component generated by another.

Vectors

Appendix A p4 §1·1 Space & Time.

In this appendix, we give a summary of the properties of vectors which are used in the text.

A.1 Definitions and Elementary Properties

A *vector* $\mathbf{a}$ is a quantity specified by a magnitude, written a or $|\mathbf{a}|$, and a direction in space. It is to be contrasted with a *scalar*, which is a quantity specified by a magnitude alone. The vector $\mathbf{a}$ may be represented geometrically by an arrow of length a drawn from any point in the appropriate direction. In particular, the position of a point P with respect to a given origin O may be specified by the *position vector* $\mathbf{r}$ drawn from O to P.

Any vector can be specified, with respect to a given set of Cartesian axes, by three components. If x, y, z are the Cartesian co-ordinates of P, then we write $\mathbf{r} = (x, y, z)$, and say that x, y, z are the components of $\mathbf{r}$. (See Fig. A.1.) We often speak of P as 'the point $\mathbf{r}$'. When P coincides with O, we have the *zero vector* $\mathbf{0} = (0, 0, 0)$ of length 0 and indeterminate direction. For a general vector $\mathbf{a}$, we write $\mathbf{a} = (a_x, a_y, a_z)$.

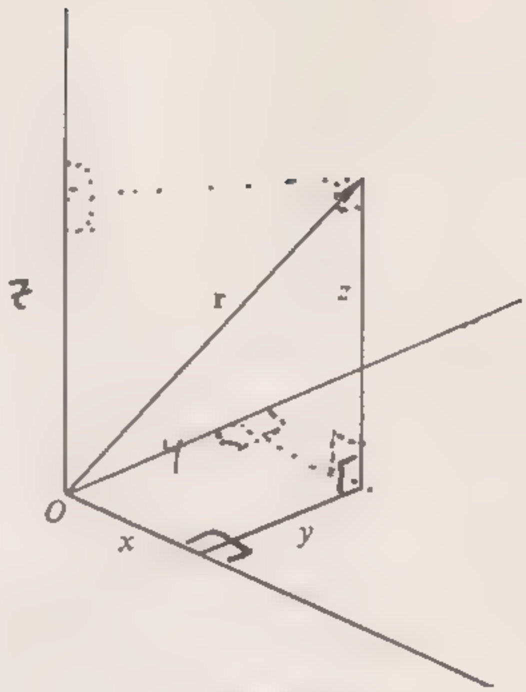


Fig. A.1

The product of a vector $\mathbf{a}$ and a scalar c is $c\mathbf{a} = (ca_x, ca_y, ca_z)$. If $c>0$, it is a vector in the same direction as $\mathbf{a}$, and of length ca; if $c<0$, it is in the opposite direction, and of length $|c|a$. In particular, if $c = 1/a$, we have the *unit vector* in the direction of $\mathbf{a}$, $\hat{\mathbf{a}} = \mathbf{a}/a$.

Addition of two vectors $\mathbf{a}$ and $\mathbf{b}$ may be defined geometrically by drawing one vector from the head of the other, as in Fig. A.2. (This is the 'parallelogram law' for addition of forces.) Subtraction is defined similarly by Fig. A.3. In terms of components,

$$\mathbf{a}+\mathbf{b} = (a_x+b_x, a_y+b_y, a_z+b_z).$$

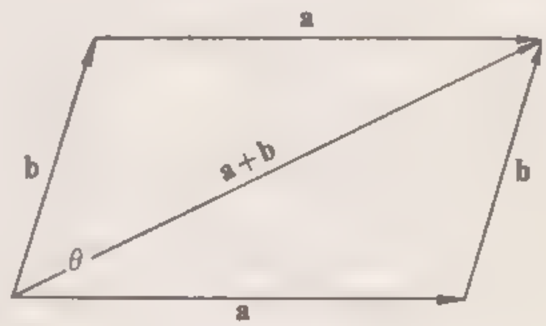


Fig. A.2

It is often useful to introduce three unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, pointing in the directions of the x-, y-, z-axes, respectively. They form what is known as an *orthonormal triad*—a set of three mutually perpendicular vectors of unit length. It is clear from Fig. A.1 that any vector $\mathbf{r}$ can be written as a sum of three vectors along the three axes,

$$\mathbf{r} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}. \tag{A.1}$$

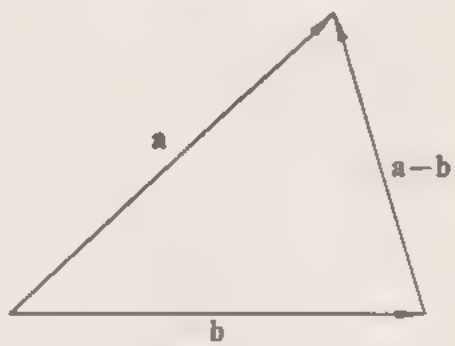


Fig. A.3

If θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, then by elementary trigonometry the length of their sum is given by

$$|\mathbf{a}+\mathbf{b}|^2 = a^2+b^2+2ab\cos\theta.$$

It is useful to define the *scalar product* $\mathbf{a}\cdot\mathbf{b}$ ('$\mathbf{a}$ *dot* $\mathbf{b}$') as

$$\mathbf{a}\cdot\mathbf{b} = ab\cos\theta. \tag{A.2}$$

Note that this is equal to the length of $\mathbf{a}$ multiplied by the projection of $\mathbf{b}$ on $\mathbf{a}$, or vice versa.

In particular, the *square* of $\mathbf{a}$ is

$$\mathbf{a}^2 = \mathbf{a}\cdot\mathbf{a} = a^2. \tag{A.3}$$

Thus we can rewrite the relation above as

$$(\mathbf{a}+\mathbf{b})^2 = \mathbf{a}^2+\mathbf{b}^2+2\mathbf{a}\cdot\mathbf{b},$$

and similarly

$$(\mathbf{a}-\mathbf{b})^2 = \mathbf{a}^2+\mathbf{b}^2-2\mathbf{a}\cdot\mathbf{b}.$$

All the ordinary rules of algebra are valid for sums and scalar products of vectors, save one. (For example, the commutative law of addition, $\mathbf{a}+\mathbf{b} = \mathbf{b}+\mathbf{a}$ is obvious from Fig. A.2, and the other laws can be deduced from appropriate figures.) The exception is the following: for two scalars, the equation $ab = 0$ implies that either $a = 0$ or $b = 0$ (or, of course, that both $= 0$), but we can find two non-zero vectors for which $\mathbf{a}\cdot\mathbf{b} = 0$. In fact, this is the case if $\theta = \frac{1}{2}\pi$, that is if the vectors are orthogonal:

$$\mathbf{a}\cdot\mathbf{b} = 0 \text{ if } \mathbf{a}\perp\mathbf{b}.$$

The scalar products of the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are

$$\begin{aligned} \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = 1,\\ \mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0. \end{aligned} \tag{A.4}$$

Thus, taking the scalar product of each in turn with (A.1), we find

$$\mathbf{i}\cdot\mathbf{r} = x, \qquad \mathbf{j}\cdot\mathbf{r} = y, \qquad \mathbf{k}\cdot\mathbf{r} = z. \tag{A.5}$$

These relations express the fact that the components of $\mathbf{r}$ are equal to its projections on the co-ordinate axes.

More generally, if we take the scalar product of two vectors $\mathbf{a}$ and $\mathbf{b}$, we find

$$\mathbf{a}\cdot\mathbf{b} = a_x b_x + a_y b_y + a_z b_z, \tag{A.6}$$

and, in particular,

$$\mathbf{r}^2 = r^2 = x^2+y^2+z^2. \tag{A.7}$$

A.2 The Vector Product

Any two nonparallel vectors $\mathbf{a}$ and $\mathbf{b}$ drawn from O define a unique axis through O perpendicular to the plane containing $\mathbf{a}$ and $\mathbf{b}$. It is useful to define the *vector product* $\mathbf{a}\wedge\mathbf{b}$ ('$\mathbf{a}$ *cross* $\mathbf{b}$', sometimes written $\mathbf{a}\times\mathbf{b}$) to be a vector along this axis whose magnitude is the area of the parallelogram with edges $\mathbf{a}$, $\mathbf{b}$,

$$|\mathbf{a}\wedge\mathbf{b}| = ab\sin\theta. \tag{A.8}$$

(See Fig. A.4.) To distinguish between the two opposite directions along the axis, we introduce a convention: the direction of $\mathbf{a}\wedge\mathbf{b}$ is that in which a right-hand screw would move when turned from $\mathbf{a}$ to $\mathbf{b}$.

A vector whose sense is merely conventional, and would be reversed by changing from a right-hand to a left-hand convention is called an *axial* vector, as opposed to an ordinary or *polar* vector. For example, velocity and force are polar vectors, but angular velocity is an axial vector (see §5.1). The vector product of two polar vectors is thus an axial vector.

The vector product has one very important, but unfamiliar, property. If we interchange $\mathbf{a}$ and $\mathbf{b}$, we reverse the sign of the vector product,

$$\mathbf{b}\wedge\mathbf{a} = -\mathbf{a}\wedge\mathbf{b}. \tag{A.9}$$

It is essential to remember this fact when manipulating any expression involving vector products. In particular, the vector product of a vector with itself is the zero vector,

$$\mathbf{a}\wedge\mathbf{a} = \mathbf{0}.$$

More generally, $\mathbf{a}\wedge\mathbf{b}$ vanishes if $\theta = 0$ or π,

$$\mathbf{a}\wedge\mathbf{b} = \mathbf{0} \quad \text{if } \mathbf{a}\|\mathbf{b}.$$

If we choose our co-ordinate axes to be right-handed, then the vector products of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are

$$\begin{gathered} \mathbf{i}\wedge\mathbf{i} = \mathbf{j}\wedge\mathbf{j} = \mathbf{k}\wedge\mathbf{k} = \mathbf{0}, \\ \mathbf{i}\wedge\mathbf{j} = \mathbf{k}, \qquad \mathbf{j}\wedge\mathbf{i} = -\mathbf{k}, \\ \mathbf{j}\wedge\mathbf{k} = \mathbf{i}, \qquad \mathbf{k}\wedge\mathbf{j} = -\mathbf{i}, \\ \mathbf{k}\wedge\mathbf{i} = \mathbf{j}, \qquad \mathbf{i}\wedge\mathbf{k} = -\mathbf{j}. \end{gathered} \tag{A.10}$$

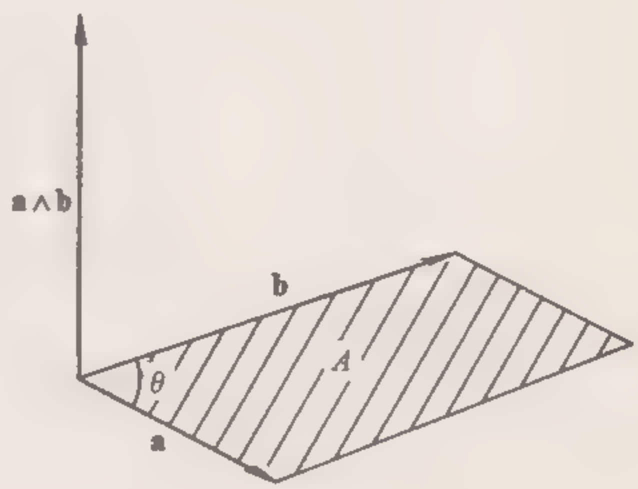


Fig. A.4

Thus, when we form the vector product of $\mathbf{a}$ and $\mathbf{b}$ we obtain

$$\mathbf{a}\wedge\mathbf{b} = \mathbf{i}(a_y b_z - a_z b_y) + \mathbf{j}(a_z b_x - a_x b_z) + \mathbf{k}(a_x b_y - a_y b_x).$$

This relation may conveniently be expressed in the form of a determinant

$$\mathbf{a}\wedge\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}. \tag{A.11}$$

From any three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, we can form the *scalar triple product* $(\mathbf{a}\wedge\mathbf{b})\cdot\mathbf{c}$. Geometrically, it represents the volume V of the parallelepiped with adjacent edges $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. (See Fig. A.5.) For, if φ is the angle between $\mathbf{c}$ and $\mathbf{a}\wedge\mathbf{b}$, then

$$(\mathbf{a}\wedge\mathbf{b})\cdot\mathbf{c} = |\mathbf{a}\wedge\mathbf{b}|c\cos\varphi = Ah = V,$$

where A is the area of the base, and $h = c\cos\varphi$ is the height. The volume is reckoned positive if $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ form a right-handed triad, and

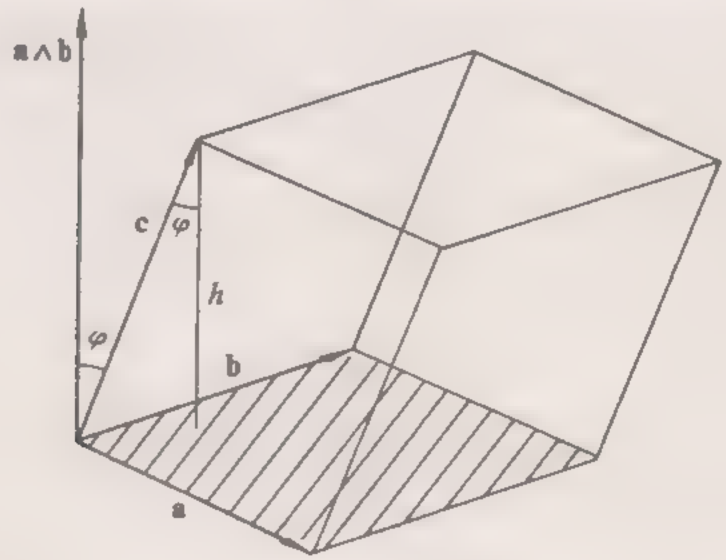


Fig. A.5

negative if they form a left-handed triad. For example, $(\mathbf{i}\wedge\mathbf{j})\cdot\mathbf{k} = 1$, but $(\mathbf{i}\wedge\mathbf{k})\cdot\mathbf{j} = -1$.

In terms of components, we can evaluate the scalar triple product by taking the scalar product of $\mathbf{c}$ with (A.11). We find

$$(\mathbf{a}\wedge\mathbf{b})\cdot\mathbf{c} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}. \tag{A.12}$$

Either from this formula, or from its geometrical interpretation, we see that the scalar triple product is unchanged by any cyclic permutation of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, but changes signs if any pair is interchanged,

$$\begin{aligned}(\mathbf{a}\wedge\mathbf{b})\cdot\mathbf{c} &= (\mathbf{b}\wedge\mathbf{c})\cdot\mathbf{a} = (\mathbf{c}\wedge\mathbf{a})\cdot\mathbf{b} \\ &= -(\mathbf{b}\wedge\mathbf{a})\cdot\mathbf{c} = -(\mathbf{c}\wedge\mathbf{b})\cdot\mathbf{a} = -(\mathbf{a}\wedge\mathbf{c})\cdot\mathbf{b}.\end{aligned} \tag{A.13}$$

Moreover, we may interchange the dot and cross,

$$(\mathbf{a}\wedge\mathbf{b})\cdot\mathbf{c} = \mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c}). \tag{A.14}$$

(For this reason, the more symmetrical notation $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ is sometimes used for the scalar triple product.)

Note that the scalar triple product vanishes if any two vectors are equal, or parallel. More generally, it vanishes if $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are coplanar.

We can also form the *vector triple product* $(\mathbf{a}\wedge\mathbf{b})\wedge\mathbf{c}$. Since this vector is perpendicular to $\mathbf{a}\wedge\mathbf{b}$, it must lie in the plane of $\mathbf{a}$ and $\mathbf{b}$, and must therefore be a linear combination of these two vectors. It is not hard to show, by writing out the components, that

$$(\mathbf{a}\wedge\mathbf{b})\wedge\mathbf{c} = \mathbf{b}(\mathbf{a}\cdot\mathbf{c})-\mathbf{a}(\mathbf{b}\cdot\mathbf{c}). \tag{A.15}$$

Similarly,

$$\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c})-\mathbf{c}(\mathbf{a}\cdot\mathbf{b}). \tag{A.16}$$

Note that these expressions are unequal, so that we cannot omit the brackets in a vector triple product. It is useful to notice that in both these formulae the term with positive sign is the middle vector $\mathbf{b}$ times the scalar product of the other two.

A.3 Differentiation and Integration of Vectors

We are often concerned with vectors which are functions of some scalar parameter, for example the position of a particle as a function of time, $\mathbf{r}(t)$. The vector distance travelled by the particle in a short time interval Δt is

$$\Delta\mathbf{r} = \mathbf{r}(t+\Delta t)-\mathbf{r}(t).$$

(See Fig. A.6.) The velocity, or derivative with respect to t, is defined just as for scalars, as the limit of a ratio,

$$\dot{\mathbf{r}} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \lim_{\Delta t\to 0}\frac{\Delta\mathbf{r}}{\Delta t}. \tag{A.17}$$

In the limit, the direction of this vector is that of the tangent to the path of the particle, and its magnitude is the speed in the usual sense. In terms of co-ordinates,

$$\dot{\mathbf{r}} = (\dot{x}, \dot{y}, \dot{z}).$$

Derivatives of other vectors are defined similarly. In particular, we can differentiate again to form the acceleration vector $\ddot{\mathbf{r}}$.

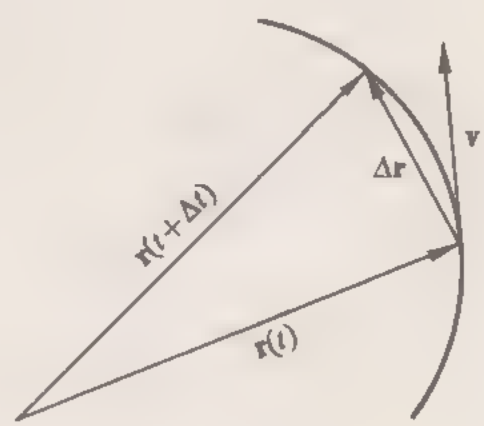


Fig. A.6

The 'operator' does not commute?

It is easy to show that all the usual rules for differentiating sums and products apply also to vectors. For example,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{a}\wedge\mathbf{b}) = \frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t}\wedge\mathbf{b}+\mathbf{a}\wedge\frac{\mathrm{d}\mathbf{b}}{\mathrm{d}t},$$

though in this case one must be careful to preserve the order of the two factors, because of the antisymmetry of the vector product.

Note that the derivative of the magnitude of $\mathbf{r}$, $\mathrm{d}r/\mathrm{d}t$, is not the same thing as the magnitude of the derivative $|\mathrm{d}\mathbf{r}/\mathrm{d}t|$. For example, for a particle moving in a circle, r is constant, so that $\dot{r} = 0$, but clearly $|\dot{\mathbf{r}}|$ is not zero in general. In fact, applying the rule for differentiating a scalar product to $\mathbf{r}^2$, we obtain

$$2r\dot{r} = \frac{\mathrm{d}}{\mathrm{d}t}(r^2) = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}^2) = 2\mathbf{r}\cdot\dot{\mathbf{r}}$$

which may also be written

$$\dot{r} = \hat{\mathbf{r}}\cdot\dot{\mathbf{r}}. \tag{A.18}$$

Thus the rate of change of the distance r from the origin is equal to the radial component of the velocity vector.

We can also define the integral of a vector. If $\mathbf{v} = \mathrm{d}\mathbf{r}/\mathrm{d}t$, then we also write

$$\mathbf{r} = \int \mathbf{v}\,\mathrm{d}t,$$

and say that $\mathbf{r}$ is the *integral* of $\mathbf{v}$. If we are given $\mathbf{v}(t)$ as a function of time, and the initial value of $\mathbf{r}$, $\mathbf{r}(t_0)$, then the position at any later time is given by the definite integral

$$\mathbf{r}(t_1) = \mathbf{r}(t_0)+\int_{t_0}^{t_1}\mathbf{v}(t)\,\mathrm{d}t. \tag{A.19}$$

This is equivalent to the three scalar equations for the components, for example

$$x(t_1) = x(t_0)+\int_{t_0}^{t_1}v_x(t)\,\mathrm{d}t.$$

One can show, exactly as for scalars, that the integral in (A.19) may be expressed as the limit of a sum.

A.4 Gradient, Divergence and Curl

There are many quantities in physics which are functions of position in space; for example, temperature, gravitational potential or electric field. Such quantities are known as *fields*. A *scalar field* is a scalar function $\phi(x, y, z)$ of position in space; a *vector field*

is a vector function $\mathbf{A}(x, y, z)$. We can also indicate the position in space by the position vector $\mathbf{r}$, and write $\phi(\mathbf{r})$ or $\mathbf{A}(\mathbf{r})$.

Now let us consider the three partial derivatives of a scalar field, $\partial\phi/\partial x$, $\partial\phi/\partial y$, $\partial\phi/\partial z$. They form the components of a vector field, known as the *gradient* of ϕ, and written grad ϕ, or $\nabla\phi$ ('*del* ϕ'). To show that they really are the components of a *vector*, we have to show that it can be defined in a manner which is independent of the choice of axes. We note that if $\mathbf{r}$ and $\mathbf{r}+\mathrm{d}\mathbf{r}$ are two neighbouring points, then the difference between the values of ϕ at these points is

$$\mathrm{d}\phi = \phi(\mathbf{r}+\mathrm{d}\mathbf{r})-\phi(\mathbf{r}) = \frac{\partial\phi}{\partial x}\,\mathrm{d}x+\frac{\partial\phi}{\partial y}\,\mathrm{d}y+\frac{\partial\phi}{\partial z}\,\mathrm{d}z = \mathrm{d}\mathbf{r}\cdot\nabla\phi. \tag{A.20}$$

Now, if the distance $|\mathrm{d}\mathbf{r}|$ is fixed, then this scalar product takes on its maximum value when $\mathrm{d}\mathbf{r}$ is in the direction of $\nabla\phi$. Hence we conclude that the direction of $\nabla\phi$ is the direction in which ϕ increases most rapidly. Moreover, its magnitude is the rate of increase of ϕ with distance in this direction. (This is the reason for the name 'gradient'.) Clearly, therefore, we could *define* $\nabla\phi$ by these properties, which are independent of any choice of axes.

We are often interested in the value of a scalar field ϕ evaluated at the position of a particle, $\phi(\mathbf{r}(t))$. From (A.20) it follows that the rate of change of $\phi(\mathbf{r}(t))$ is

$$\dot{\phi}(\mathbf{r}(t)) = \dot{\mathbf{r}}\cdot\nabla\phi. \tag{A.21}$$

The symbol ∇ may be regarded as a vector which is also a differential operator (like $\mathrm{d}/\mathrm{d}x$), given by

$$\nabla = \mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}. \tag{A.22}$$

We can also apply it to a vector field $\mathbf{A}$. The *divergence* of $\mathbf{A}$ is defined to be

$$\operatorname{div}\mathbf{A} = \nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}, \tag{A.23}$$

and the *curl* of $\mathbf{A}$ to be*

$$\operatorname{curl}\mathbf{A} = \nabla\wedge\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}. \tag{A.24}$$

* Instead of curl $\mathbf{A}$, the alternative notation rot $\mathbf{A}$ is sometimes used.

This latter expression is an abbreviation for the expanded form

$$\mathbf{A} = \mathbf{i}\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)+\mathbf{j}\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)+\mathbf{k}\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right).$$

In particular, we may take $\mathbf{A}$ to be the gradient of a scalar field, $\mathbf{A} = \nabla\phi$. Then its divergence is called the *Laplacian* of ϕ,

$$\nabla\cdot\nabla\phi = \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}. \tag{A.25}$$

Just as $\mathbf{a}\wedge\mathbf{a} = \mathbf{0}$, we find that the curl of a gradient vanishes,

$$\nabla\wedge\nabla\phi = \mathbf{0}. \tag{A.26}$$

For example, its z component is

$$\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y}\right)-\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}\right) = 0.$$

Similarly, one can show that the divergence of a curl vanishes,

$$\nabla\cdot(\nabla\wedge\mathbf{A}) = 0. \tag{A.27}$$

The rule for differentiating products can also be applied to expressions involving ∇. For example, $\nabla\cdot(\mathbf{A}\wedge\mathbf{B})$ is a sum of two terms, in one of which ∇ acts on $\mathbf{A}$ only, and in the other on $\mathbf{B}$ only. The gradient of a product of scalar fields can be written

$$\nabla(\phi\psi) = \psi\nabla\phi+\phi\nabla\psi,$$

but, when vector fields are involved, we have to remember that the order of the factors as a product of vectors cannot be changed without affecting the signs. Thus we have

$$\nabla\cdot(\mathbf{A}\wedge\mathbf{B}) = \mathbf{B}\cdot(\nabla\wedge\mathbf{A})-\mathbf{A}\cdot(\nabla\wedge\mathbf{B}),$$

and similarly

$$\nabla\wedge(\phi\mathbf{A}) = \phi(\nabla\wedge\mathbf{A})-\mathbf{A}\wedge(\nabla\phi).$$

An important identity, analogous to the expansion of the vector triple product (A.16) is

$$\nabla\wedge(\nabla\wedge\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}, \tag{A.28}$$

where of course

$$\nabla^2\mathbf{A} = \frac{\partial^2\mathbf{A}}{\partial x^2}+\frac{\partial^2\mathbf{A}}{\partial y^2}+\frac{\partial^2\mathbf{A}}{\partial z^2}.$$

It may easily be proved by inserting the expressions in terms of components.

A.5 Integral Theorems

There are three important theorems for vectors which are generalizations of the fundamental theorem of the calculus,

$$\int_{x_0}^{x_1} \frac{df}{dx}\,dx = f(x_1)-f(x_0).$$

First, consider a curve C in space, running from $\mathbf{r}_0$ to $\mathbf{r}_1$. (See Fig. A.7.) Let the directed element of length along C be $d\mathbf{r}$. If ϕ is a scalar field, then, according to (A.20), the change in ϕ along this element of length is $d\phi = d\mathbf{r}\cdot\nabla\phi$. Thus, integrating from $\mathbf{r}_0$ to $\mathbf{r}_1$, we obtain the first of the integral theorems,

$$\int_{\mathbf{r}_0}^{\mathbf{r}_1} d\mathbf{r}\cdot\nabla\phi = \phi(\mathbf{r}_1)-\phi(\mathbf{r}_0). \tag{A.29}$$

The integral on the left is called the *line integral* of $\nabla\phi$ along C.

This theorem may be used to relate the potential energy function $V(\mathbf{r})$ for a conservative force to the work done in going from some fixed point $\mathbf{r}_0$, where V is chosen to vanish, to $\mathbf{r}$. Thus, if $\mathbf{F} = -\nabla V$, then

$$V(\mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} d\mathbf{r}\cdot\mathbf{F}. \tag{A.30}$$

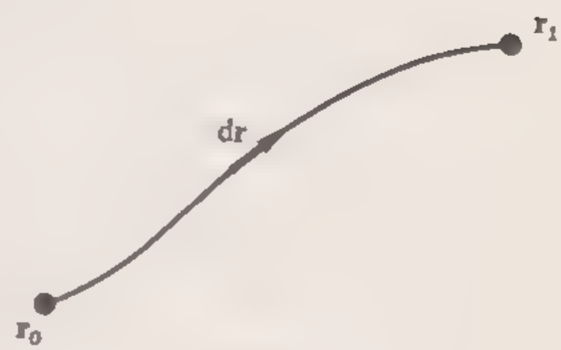


Fig. A.7

When $\mathbf{F}$ is conservative, this integral depends only on its end-points, and not on the path C chosen between them. Conversely, if this condition is satisfied, we can define V by (A.30), and the force must be conservative. The condition that two line integrals of the form (A.30) should be equal whenever their end-points coincide may be restated by saying that the line integral round any *closed* path should vanish. Physically, this means that no work is done in taking the particle round a loop which returns to its starting point. The integral round a closed loop C is usually denoted by the symbol $\oint_C$. Thus we require

$$\oint_C d\mathbf{r}\cdot\mathbf{F} = 0 \tag{A.31}$$

for all closed loops C.

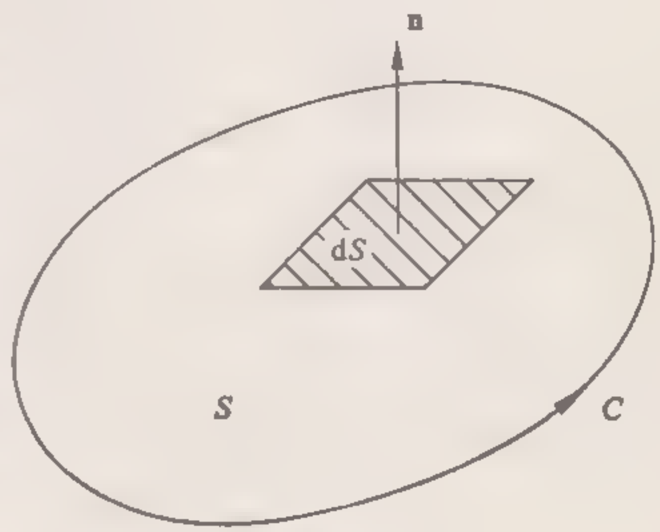


Fig. A.8

This condition may be simplified by using the second of the integral theorems—Stokes' theorem. Consider a curved surface S, bounded by the closed curve C. If one side of S is chosen to be the 'positive' side, then the positive direction round C may be defined by the right-hand screw convention. (See Fig. A.8.) Take a small element of the surface, of area dS, and let $\mathbf{n}$ be the unit vector normal to the

element, and directed towards its positive side. Then the directed element of area is defined to be $d\mathbf{S} = \mathbf{n}dS$. Stokes' theorem states that if $\mathbf{A}$ is any vector field, then

$$\iint_S d\mathbf{S}\cdot(\nabla\wedge\mathbf{A}) = \oint_C d\mathbf{r}\cdot\mathbf{A}. \tag{A.32}$$

The application of this theorem to (A.31) is immediate. If the line integral round C is required to vanish for all closed curves C, then the surface integral must vanish for all surfaces S. But this is only possible if the integrand vanishes identically. So the condition for a force $\mathbf{F}$ to be conservative is

$$\nabla\wedge\mathbf{F} = \mathbf{0}. \tag{A.33}$$

We shall not prove Stokes' theorem. However, it is easy to verify that it is true for a small rectangular surface. Suppose S is a rectangle in the xy-plane of area $dxdy$. Then $d\mathbf{S} = \mathbf{k}dxdy$, so the surface integral is

$$\mathbf{k}\cdot(\nabla\wedge\mathbf{A})\,dxdy = \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)dxdy. \tag{A.34}$$

The line integral involves four terms, one from each edge. The two terms arising from the edges parallel to the x-axis involve the x component of $\mathbf{A}$ evaluated for different values of y. They therefore contribute

$$A_x(y)\,dx - A_x(y+dy)\,dx = -\frac{\partial A_x}{\partial y}\,dxdy.$$

Similarly, the other pair of edges yield the first term of (A.34).

We can also find a necessary and sufficient condition for a field $\mathbf{B}(\mathbf{r})$ to have the form of a curl,

$$\mathbf{B} = \nabla\wedge\mathbf{A}.$$

By (A.27), such a field must satisfy

$$\nabla\cdot\mathbf{B} = 0. \tag{A.35}$$

The proof that this is also a sufficient condition (which we shall not give in detail) follows much the same lines as before. One can show that it is sufficient that the surface integral of $\mathbf{B}$ over any closed surface should vanish,

$$\iint_S d\mathbf{S}\cdot\mathbf{B} = 0,$$

and then use the third of the integral theorems, Gauss' theorem. This states that if V is a volume in space bounded by the closed surface S, then for any vector field $\mathbf{B}$,

$$\iiint_V \mathrm{d}V\, \nabla\cdot\mathbf{B} = \iint_S \mathrm{d}\mathbf{S}\cdot\mathbf{B}, \tag{A.36}$$

where $\mathrm{d}V$ denotes the volume element $\mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z$, and the positive side of S is taken to be the outside.

It is again easy to verify Gauss' theorem for a small rectangular volume $\mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z$. The volume integral is

$$\left(\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z. \tag{A.37}$$

The surface integral consists of six terms, one for each face. Consider the faces parallel to the xy-plane, with directed surface elements $\mathbf{k}\,\mathrm{d}x\mathrm{d}y$ and $-\mathbf{k}\,\mathrm{d}x\mathrm{d}y$. Their contributions involve $\mathbf{k}\cdot\mathbf{B} = B_z$ evaluated for different values of z. Thus they contribute

$$B_z(z+\mathrm{d}z)\,\mathrm{d}x\mathrm{d}y - B_z(z)\,\mathrm{d}x\mathrm{d}y = \frac{\partial B_z}{\partial z}\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

Similarly, the other terms of (A.37) come from the other faces.

A.6 Curvilinear Co-ordinates

One of the uses of the integral theorems is to provide expressions for the gradient, divergence and curl in terms of curvilinear co-ordinates.

Consider a set of orthogonal curvilinear co-ordinates q_1, q_2, q_3, and denote the elements of length along the three co-ordinate curves by $h_1\mathrm{d}q_1$, $h_2\mathrm{d}q_2$, $h_3\mathrm{d}q_3$. For example, in cylindrical polars,

$$h_\rho = 1, \qquad h_\varphi = \rho, \qquad h_z = 1, \tag{A.38}$$

and in spherical polars

$$h_r = 1, \qquad h_\theta = r, \qquad h_\varphi = r\sin\theta. \tag{A.39}$$

Now consider a scalar field ψ, and two neighbouring points (q_1, q_2, q_3) and $(q_1, q_2, q_3+\mathrm{d}q_3)$. Then the difference between the values of ψ at these points is

$$\frac{\partial\psi}{\partial q_3}\mathrm{d}q_3 = \mathrm{d}\psi = \mathrm{d}\mathbf{r}\cdot\nabla\psi = h_3\mathrm{d}q_3(\nabla\psi)_3,$$

where $(\nabla\psi)_3$ is the component of $\nabla\psi$ in the direction of increasing q_3. Hence we find

$$(\nabla\psi)_3 = \frac{1}{h_3}\frac{\partial\psi}{\partial q_3}, \tag{A.40}$$

with similar expressions for the other components. Thus, in cylindrical and spherical polars, we have

$$\nabla\psi = \left(\frac{\partial\psi}{\partial\rho}, \frac{1}{\rho}\frac{\partial\psi}{\partial\varphi}, \frac{\partial\psi}{\partial z}\right), \tag{A.41}$$

and

$$\nabla\psi = \left(\frac{\partial\psi}{\partial r}, \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\varphi}\right). \tag{A.42}$$

To find an expression for the divergence, we use Gauss' theorem, applied to a small volume bounded by the co-ordinate surfaces. The volume integral is

$$(\nabla\cdot\mathbf{A})h_1\mathrm{d}q_1h_2\mathrm{d}q_2h_3\mathrm{d}q_3.$$

In the surface integral, the terms arising from the faces which are surfaces of constant q_3 are of the form $A_3h_1\mathrm{d}q_1h_2\mathrm{d}q_2$, evaluated for two different values of q_3. They therefore contribute

$$\frac{\partial}{\partial q_3}(h_1h_2A_3)\,\mathrm{d}q_1\mathrm{d}q_2\mathrm{d}q_3.$$

Adding the terms from all three pairs of faces, and comparing with the volume integral, we obtain

$$\nabla\cdot\mathbf{A} = \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial q_1}(h_2h_3A_1)+\frac{\partial}{\partial q_2}(h_3h_1A_2)+\frac{\partial}{\partial q_3}(h_1h_2A_3)\right\}. \tag{A.43}$$

In particular, in cylindrical and spherical polars,

$$\nabla\cdot\mathbf{A} = \frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial\rho}+\frac{1}{\rho}\frac{\partial A_\varphi}{\partial\varphi}+\frac{\partial A_z}{\partial z}, \tag{A.44}$$

and

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial(r^2A_r)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}. \tag{A.45}$$

To find the curl, we use Stokes' theorem in a similar way. If we consider a small element of a surface $q_3 =$ constant, bounded by curves of constant q_1 and q_2, then the surface integral is

$$(\nabla\wedge\mathbf{A})_3h_1\mathrm{d}q_1h_2\mathrm{d}q_2.$$

In the line integral round the boundary, the two edges of constant q_2 involve $A_1 h_1 \mathrm{d}q_1$ evaluated for different values of q_2, and contribute

$$-\frac{\partial}{\partial q_2}(h_1 A_1)\,\mathrm{d}q_1 \mathrm{d}q_2.$$

Hence, adding the contribution from the other two edges, we obtain

$$(\nabla \wedge \mathbf{A})_3 = \frac{1}{h_1 h_2}\left\{\frac{\partial}{\partial q_1}(h_2 A_2) - \frac{\partial}{\partial q_2}(h_1 A_1)\right\}, \tag{A.46}$$

with similar expressions for the other components.

Thus, in particular, in cylindrical polars,

$$\nabla \wedge \mathbf{A} = \left\{\frac{1}{\rho}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z},\ \frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho},\ \frac{1}{\rho}\left(\frac{\partial(\rho A_\varphi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \varphi}\right)\right\}, \tag{A.47}$$

and in spherical polars

$$\nabla \wedge \mathbf{A} = \left\{\frac{1}{r \sin\theta}\left(\frac{\partial(\sin\theta A_\varphi)}{\partial \theta} - \frac{\partial A_\theta}{\partial \varphi}\right),\right.$$
$$\left.\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial \varphi} - \frac{1}{r}\frac{\partial(r A_\varphi)}{\partial r},\ \frac{1}{r}\left(\frac{\partial(r A_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right)\right\}. \tag{A.48}$$

Finally, combining the expressions for the divergence and gradient, we can find the Laplacian of a scalar field. It is

$$\nabla^2 \psi = \frac{1}{h_1 h_2 h_3}\left\{\frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial \psi}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial \psi}{\partial q_2}\right)\right.$$
$$\left. + \frac{\partial}{\partial q_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial \psi}{\partial q_3}\right)\right\}. \tag{A.49}$$

In cylindrical polars,

$$\nabla^2 \psi = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \psi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \psi}{\partial \varphi^2} + \frac{\partial^2 \psi}{\partial z^2}, \tag{A.50}$$

and, in spherical polars,

$$\nabla^2 \psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \psi}{\partial \theta}\right)$$
$$+ \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \psi}{\partial \varphi^2}. \tag{A.51}$$

PROBLEMS

1 By drawing appropriate figures, prove the following laws of vector algebra:

$$(\mathbf{a}+\mathbf{b})+\mathbf{c} = \mathbf{a}+(\mathbf{b}+\mathbf{c}),$$
$$\lambda(\mathbf{a}+\mathbf{b}) = \lambda\mathbf{a}+\lambda\mathbf{b},$$
$$\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}.$$

(Note that $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ need not be coplanar.)

2 Show that $(\mathbf{a} \wedge \mathbf{b})\cdot(\mathbf{c} \wedge \mathbf{d}) = \mathbf{a}\cdot\mathbf{c}\, \mathbf{b}\cdot\mathbf{d}-\mathbf{a}\cdot\mathbf{d}\, \mathbf{b}\cdot\mathbf{c}$.

3 Evaluate $\nabla \wedge (\mathbf{a} \wedge \mathbf{b})$.

4 Prove that $\mathbf{a} \wedge (\mathbf{b}+\mathbf{c}) = \mathbf{a} \wedge \mathbf{b}+\mathbf{a} \wedge \mathbf{c}$. (Hint: Show first that in $\mathbf{a} \wedge \mathbf{b}$, $\mathbf{b}$ may be replaced by its projection on the plane normal to $\mathbf{a}$, and then prove the result for vectors in this plane.)

5 Evaluate the components of $\nabla^2\mathbf{A}$ in cylindrical polar co-ordinates using the identity (A.28). Show that they are not the same as the scalar Laplacians of the components of $\mathbf{A}$.

The Electromagnetic Field

Appendix B

Electromagnetic theory lies outside the scope of this book. However, since we have discussed various examples involving electromagnetic fields, it may be useful to summarize some relevant properties of these fields here. We shall simply quote the results without proof, and we shall not consider the case of dielectric or magnetic media. We shall use Gaussian units, but quote the forms appropriate to SI units in brackets.

The basic equations of electromagnetic theory are Maxwell's equations. In the absence of dielectric or magnetic media, they may be expressed in terms of two fields, the electric field **E** and the magnetic field **B**. There are two equations involving these fields alone,

$$\nabla \wedge \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = \mathbf{0}, \qquad \left[\nabla \wedge \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0}\right] \tag{B.1}$$

$$\nabla \cdot \mathbf{B} = 0, \qquad [\nabla \cdot \mathbf{B} = 0] \tag{B.2}$$

and two more involving also the electric charge density ρ and current density **j**,

$$\nabla \wedge \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{j}, \qquad \left[\mu_0^{-1}\nabla \wedge \mathbf{B} - \varepsilon_0\frac{\partial \mathbf{E}}{\partial t} = \mathbf{j}\right] \tag{B.3}$$

$$\nabla \cdot \mathbf{E} = 4\pi\rho. \qquad [\varepsilon_0 \nabla \cdot \mathbf{E} = \rho] \tag{B.4}$$

The basic set of equations is completed by the Lorentz force equation, which determines the force on a particle of charge q moving with velocity **v**,

$$\mathbf{F} = q\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \wedge \mathbf{B}\right). \qquad [\mathbf{F} = q(\mathbf{E} + \mathbf{v} \wedge \mathbf{B})] \tag{B.5}$$

From (B.2), it follows that there must exist a *vector potential* **A** such that

$$\mathbf{B} = \nabla \wedge \mathbf{A}. \tag{B.6}$$

Substituting in (B.1), we then find that there must exist a *scalar potential* ϕ such that

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}. \qquad \left[\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}\right] \tag{B.7}$$

These potentials are not unique. If Λ is any scalar field, then

$$\phi' = \phi + \frac{1}{c}\frac{\partial \Lambda}{\partial t}, \qquad \left[\phi' = \phi + \frac{\partial \Lambda}{\partial t}\right]$$

$$\mathbf{A}' = \mathbf{A} - \nabla\Lambda \tag{B.8}$$

define the same fields $\mathbf{E}$ and $\mathbf{B}$ as do ϕ and $\mathbf{A}$. The transformation (B.8) is called a *gauge transformation.* In particular, we can always choose Λ so that the new potentials obey the *Lorentz gauge* condition

$$\frac{1}{c}\frac{\partial \phi'}{\partial t} + \nabla\cdot\mathbf{A}' = 0. \qquad \left[\frac{1}{c^2}\frac{\partial \phi'}{\partial t} + \nabla\cdot\mathbf{A}' = 0\right]^* \tag{B.9}$$

It is only necessary to choose Λ to be a solution of

$$\frac{1}{c^2}\frac{\partial^2 \Lambda}{\partial t^2} - \nabla^2\Lambda = -\left(\frac{1}{c}\frac{\partial \phi}{\partial t} + \nabla\cdot\mathbf{A}\right). \qquad \left[= -\left(\frac{1}{c^2}\frac{\partial \phi}{\partial t} + \nabla\cdot\mathbf{A}\right)\right]$$

When the Lorentz gauge condition is satisfied, we find from (B.3). (B.4) and the identity (A.28) that the potentials satisfy

$$\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \nabla^2\phi = 4\pi\rho, \qquad [= \varepsilon_0^{-1}\rho] \tag{B.10}$$

and

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2\mathbf{A} = \frac{4\pi}{c}\mathbf{j}. \qquad [= \mu_0\mathbf{j}] \tag{B.11}$$

When there is no electric charge or current density, these are three-dimensional wave equations, which describe a wave propagating with velocity c.

For the static case, in which all the fields are time-independent, Maxwell's equations separate into a pair of electrostatic equations,

$$\nabla\wedge\mathbf{E} = \mathbf{0}, \qquad \nabla\cdot\mathbf{E} = 4\pi\rho, \qquad [=\varepsilon_0^{-1}\rho] \tag{B.12}$$

identical with (6.46) and (6.47), and a pair of magnetostatic equations,

$$\nabla\cdot\mathbf{B} = 0, \qquad \nabla\wedge\mathbf{B} = \frac{4\pi}{c}\mathbf{j}. \qquad [=\mu_0\mathbf{j}] \tag{B.13}$$

Equation (B.10) reduces to Poisson's equation (6.48), and (B.11) expresses the vector potential similarly in terms of the current

* Note that $1/c^2 = \varepsilon_0\mu_0$.

density. The solution of (B.11) for this case is similar to (6.15), namely

$$\mathbf{A}(\mathbf{r}) = \frac{1}{c}\iiint \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}^3\mathbf{r}'. \tag{B.14}$$

[Here and below the SI form is obtained by the replacement $1/c \to \mu_0/4\pi$.] Thus, given a static distribution of charges and currents, we can calculate explicitly the scalar and vector potentials, and hence find the fields **E** and **B**.

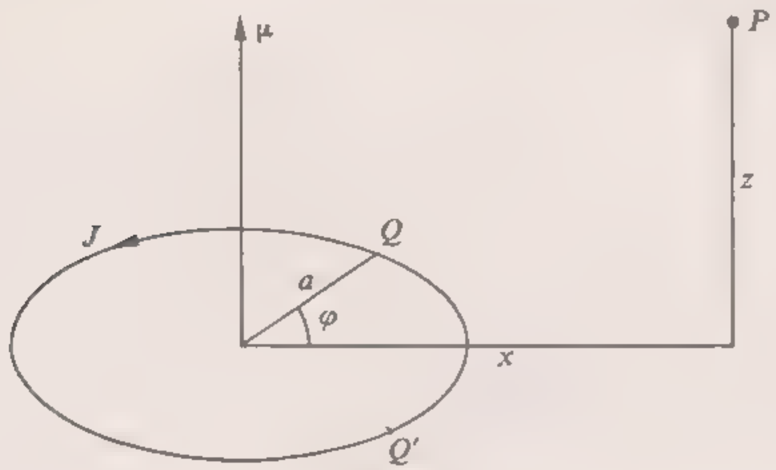


Fig. B.1

As a simple example, we consider a circular current loop of radius a in the xy-plane, carrying a current J. The equation (B.14) then reduces to a single integration round the loop,

$$\mathbf{A}(\mathbf{r}) = \frac{J}{c}\oint \frac{\mathrm{d}\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}. \tag{B.15}$$

The evaluation of this integral is much simplified by considerations of symmetry. Since the current lies in the xy-plane, A_z is clearly zero. Now let us consider a point P with co-ordinates $(x, 0, z)$. (See Fig. B.1.) For each point Q on the loop, there will be another point Q', equidistant from P. The contributions of small elements of the loop at Q and Q' to the component A_x will cancel. Thus the only non-zero component at P is A_y. Its value is

$$A_y = \frac{J}{c}\int_0^{2\pi} \frac{a\cos\varphi\,\mathrm{d}\varphi}{(r^2+a^2-2ax\cos\varphi)^{1/2}}.$$

Now we shall assume that the loop is small, so that $a \ll r$. Then the denominator is approximately

$$(r^2-2ax\cos\varphi)^{-1/2} \approx \frac{1}{r}\left(1+\frac{ax}{r^2}\cos\varphi\right)$$

whence

$$A_y = \frac{J}{cr}\int_0^{2\pi}\left(a\cos\varphi + \frac{a^2 x}{r^2}\cos^2\varphi\right)\mathrm{d}\varphi = \frac{\pi J a^2 x}{cr^3}.$$

It is clear that at an arbitrary point the only non-vanishing component of **A** will be in the φ direction of polar co-ordinates. If we define the *magnetic moment* μ of the loop to be

$$\mu = \frac{\pi a^2 J}{c}, \qquad [\mu = \pi a^2 J] \tag{B.16}$$

then the vector potential is

$$A_r = 0, \qquad A_\theta = 0, \qquad A_\varphi = \frac{\mu \sin\theta}{r^2}. \tag{B.17}$$

[Here and below the SI form is obtained by $\mu \to \mu_0\mu/4\pi$.] The corresponding magnetic field is easily evaluated using (A.48). It is

$$B_r = \frac{2\mu\cos\theta}{r^3}, \qquad B_\theta = \frac{\mu\sin\theta}{r^3}, \qquad B_\varphi = 0. \tag{B.18}$$

This is a *magnetic dipole* field. It has precisely the same form as the electric dipole field (6.11).

PROBLEM

1 Calculate the vector potential due to a short segment of wire of directed length $\mathrm{d}\mathbf{s}$, carrying a current J, placed at the origin. Evaluate the corresponding magnetic field. Find the force on another segment of length $\mathrm{d}\mathbf{s}'$, carrying current J', at $\mathbf{r}$. Show that this force does not satisfy Newton's third law. (To compute the force, treat the current element as a collection of moving charges.)

Tensors

Appendix C

From §9.4 pg 143

Scalars and vectors are the first two members of a family of quantities known as *tensors*, and described by 1, 3, 9, 27, ... components. Scalars and vectors are called tensors of rank 0, and of rank 1, respectively. In this appendix, we shall be concerned with the next member of the family, the tensors of rank 2, often called *dyadics*. We shall use the word *tensor* in this restricted sense, to mean a tensor of rank 2.*

C.1 Elementary Properties; The Dot Product

Tensors occur most frequently when one vector **b** is defined as a linear function of another vector **a**, according to

$$\begin{aligned} b_x &= T_{xx}a_x + T_{xy}a_y + T_{xz}a_z, \\ b_y &= T_{yx}a_x + T_{yy}a_y + T_{yz}a_z, \\ b_z &= T_{zx}a_x + T_{zy}a_y + T_{zz}a_z. \end{aligned} \tag{C.1}$$

We have already encountered one set of equations of this type—the relations (9.17) between the angular velocity $\boldsymbol{\omega}$ and angular momentum **J** of a rigid body.

It will be convenient to introduce a slight change of notation. We write a_1, a_2, a_3 in place of a_x, a_y, a_z, so that (C.1) may be written

$$b_i = \sum_j T_{ij}a_j, \tag{C.2}$$

where i and j run over 1, 2, 3. In this notation, the scalar product of two vectors is

$$\mathbf{a}\cdot\mathbf{b} = \sum_i a_i b_i. \tag{C.3}$$

Tensors are commonly denoted by sans-serif capitals, like T. The nine components of a tensor T may conveniently be exhibited in a square array, or *matrix*

$$\mathsf{T} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}. \tag{C.4}$$

Note that the first subscript labels the rows, and the second, the columns.

* Conventionally, the term *tensor* is used when the components are denoted by subscripts, while the name *dyadic* implies the use of the extended vector notation introduced below. The distinction is not, however, of great importance.

In view of the similarity between the expressions (C.2) and (C.3), it is natural to extend the dot notation, and write (C.2) in the form

$$\mathbf{b} = \mathbf{T}\cdot\mathbf{a}.$$

For instance, the relation (9.17) may be written

$$\mathbf{J} = \mathbf{I}\cdot\boldsymbol{\omega}$$

where $\mathbf{I}$ is the inertia tensor.

We can then form the scalar product of this vector with another vector $\mathbf{c}$, and obtain a scalar

$$\mathbf{c}\cdot\mathbf{T}\cdot\mathbf{a} = \sum_i \sum_j c_i T_{ij} a_j. \tag{C.5}$$

For example, it follows from (9.22) that the kinetic energy of a rigid body is

$$T = \tfrac{1}{2}\boldsymbol{\omega}\cdot\mathbf{I}\cdot\boldsymbol{\omega} = \tfrac{1}{2}\sum_i \sum_j \omega_i I_{ij} \omega_j.$$

For any tensor $\mathbf{T}$, we define the *transposed* tensor $\tilde{\mathbf{T}}$ by

$$\tilde{T}_{ij} = T_{ji}.$$

This corresponds to reflecting the array (C.4) in the leading diagonal. From (C.5) we see that in general $\mathbf{c}\cdot\mathbf{T}\cdot\mathbf{a}$ is not the same as $\mathbf{a}\cdot\mathbf{T}\cdot\mathbf{c}$. In fact,

$$\mathbf{a}\cdot\mathbf{T}\cdot\mathbf{c} = \sum_i \sum_j a_i T_{ij} c_j = \sum_j \sum_i c_j \tilde{T}_{ji} a_i,$$

so that

$$\mathbf{a}\cdot\mathbf{T}\cdot\mathbf{c} = \mathbf{c}\cdot\tilde{\mathbf{T}}\cdot\mathbf{a}. \tag{C.6}$$

Note that the dot always corresponds to a sum over adjacent subscripts.

The tensor $\mathbf{T}$ is called *symmetric if* $\tilde{\mathbf{T}} = \mathbf{T}$, or, equivalently, $T_{ji} = T_{ij}$. In this case, the array (C.4) is unchanged by reflection in the leading diagonal. An equivalent condition is that, for all vectors $\mathbf{a}$ and $\mathbf{c}$,

$$\mathbf{a}\cdot\mathbf{T}\cdot\mathbf{c} = \mathbf{c}\cdot\mathbf{T}\cdot\mathbf{a}. \tag{C.7}$$

Similarly, $\mathbf{T}$ is called *antisymmetric* (or *skew-symmetric*) if $\tilde{\mathbf{T}} = -\mathbf{T}$, or $T_{ji} = -T_{ij}$. For example, consider the relation giving the velocity of a point in a rotating body,

$$\mathbf{v} = \boldsymbol{\omega} \wedge \mathbf{r}.$$

This is a linear relation between the components of $\mathbf{r}$ and $\mathbf{v}$, and can therefore be written in the form

$$\mathbf{v} = \mathbf{T}\cdot\mathbf{r},$$

where $\mathbf{T}$ is some suitable tensor. It is easy to see that its components are given by

$$\mathbf{T} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}. \tag{C.8}$$

This tensor is clearly antisymmetric. Note that its diagonal elements T_{ii} are necessarily zero. In fact, any antisymmetric tensor may be associated with an axial vector in this way, and vice versa.

There is a special tensor $\mathbf{1}$ called the *unit tensor*, or *identity tensor*, which has the property that

$$\mathbf{1}\cdot\mathbf{a} = \mathbf{a} \tag{C.9}$$

for all vectors $\mathbf{a}$. Its components are

$$I_{ij} = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

or, written out in detail,

$$\mathbf{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{C.10}$$

C.2 Sums and Products; The Tensor Product

The sum of two tensors may be defined in an obvious way. The tensor $\mathbf{R} = \alpha\mathbf{S}+\beta\mathbf{T}$ is the tensor with components $R_{ij} = \alpha S_{ij}+\beta T_{ij}$. Its effect on a vector $\mathbf{a}$ is given by

$$\mathbf{R}\cdot\mathbf{a} = \alpha(\mathbf{S}\cdot\mathbf{a})+\beta(\mathbf{T}\cdot\mathbf{a}).$$

For example, it is easy to show that any tensor $\mathbf{T}$ can be written as a sum of a symmetric tensor $\mathbf{S}$ and an antisymmetric tensor $\mathbf{A}$. In fact, $\mathbf{T} = \mathbf{S}+\mathbf{A}$, where $\mathbf{S} = \frac{1}{2}(\mathbf{T}+\tilde{\mathbf{T}})$ and $\mathbf{A} = \frac{1}{2}(\mathbf{T}-\tilde{\mathbf{T}})$.

We can also define the dot product of two tensors, $\mathbf{S}\cdot\mathbf{T}$. If $\mathbf{c} = \mathbf{S}\cdot\mathbf{b}$ and $\mathbf{b} = \mathbf{T}\cdot\mathbf{a}$, then it is natural to write $\mathbf{c} = \mathbf{S}\cdot(\mathbf{T}\cdot\mathbf{a}) = (\mathbf{S}\cdot\mathbf{T})\cdot\mathbf{a}$. In terms of components,

$$c_i = \sum_j S_{ij}\Big(\sum_k T_{jk}a_k\Big) = \sum_k\Big(\sum_j S_{ij}T_{jk}\Big)a_k.$$

Hence $\mathbf{S}\cdot\mathbf{T} = \mathbf{R}$ is the tensor with components

$$R_{ik} = \sum_j S_{ij}T_{jk}. \tag{C.11}$$

Once again, the dot signifies summation over adjacent subscripts. Note the rule for constructing the elements of the product: to form the element in the ith row and kth column of $\mathbf{S}\cdot\mathbf{T}$, we take the ith row of $\mathbf{S}$, and the kth column of $\mathbf{T}$, multiply the corresponding elements, and sum. (This is known as the rule of *matrix multiplication.*)

It is important to realize that, in general, $\mathbf{T}\cdot\mathbf{S} \neq \mathbf{S}\cdot\mathbf{T}$. In fact, $\mathbf{T}\cdot\mathbf{S} = \mathbf{Q}$ has components

$$Q_{ik} = \sum_j T_{ij}S_{jk}.$$

There is one special case in which these products are equal, namely the case $\mathbf{S} = \mathbf{1}$. It is easy to see that

$$\mathbf{1}\cdot\mathbf{T} = \mathbf{T}\cdot\mathbf{1} = \mathbf{T},$$

so that $\mathbf{1}$ plays exactly the same role as the unit in ordinary algebra.

From any two vectors $\mathbf{a}$ and $\mathbf{b}$ we can form a tensor $\mathbf{T}$ whose components are $T_{ij} = a_i b_j$. This tensor is written $\mathbf{T} = \mathbf{ab}$, with no dot or cross, and is called the *tensor product* or *dyadic product* of $\mathbf{a}$ and $\mathbf{b}$. Note that

$$\mathbf{T}\cdot\mathbf{c} = (\mathbf{ab})\cdot\mathbf{c} = \mathbf{a}(\mathbf{b}\cdot\mathbf{c}),$$

so that the brackets are in fact unnecessary. The use of the tensor product allows us to write some earlier results in a different way. For example, for any vector $\mathbf{a}$,

$$\begin{aligned} \mathbf{1}\cdot\mathbf{a} = \mathbf{a} &= \mathbf{i}(\mathbf{i}\cdot\mathbf{a})+\mathbf{j}(\mathbf{j}\cdot\mathbf{a})+\mathbf{k}(\mathbf{k}\cdot\mathbf{a}) \\ &= (\mathbf{ii}+\mathbf{jj}+\mathbf{kk})\cdot\mathbf{a}, \end{aligned}$$

so that

$$\mathbf{ii}+\mathbf{jj}+\mathbf{kk} = \mathbf{1}, \tag{C.12}$$

as may easily be verified by writing out the components.

Similarly, we may write (9.16) in the form

$$\mathbf{J} = \sum m(r^2\boldsymbol{\omega}-\mathbf{rr}\cdot\boldsymbol{\omega}) = \mathbf{I}\cdot\boldsymbol{\omega},$$

where the inertia tensor is given explicitly by

$$\mathbf{I} = \sum m(r^2\mathbf{1}-\mathbf{rr}). \tag{C.13}$$

It is easy to check that the nine components of this equation reproduce (9.15).

It is clear that if $\mathbf{T} = \mathbf{ab}$, then $\tilde{\mathbf{T}} = \mathbf{ba}$. In particular, it follows that the tensor (C.13) is symmetric.

C.3 Eigenvalues; Diagonalization of a Symmetric Tensor

Throughout this section, we consider a given symmetric tensor $\mathbf{T}$. A vector $\mathbf{a}$ is called an *eigenvector* of $\mathbf{T}$ if

$$\mathbf{T}\cdot\mathbf{a} = \lambda\mathbf{a}, \tag{C.14}$$

where λ is a number called the *eigenvalue*. Equivalently, the equation (C.14) may be written

$$(\mathbf{T}-\lambda\mathbf{1})\cdot\mathbf{a} = \mathbf{0},$$

or, written out in full, with the role of a given tensor of rank 2

$$\begin{aligned}(T_{11}-\lambda)a_1+T_{12}a_2+T_{13}a_3 &= 0,\\ T_{21}a_1+(T_{22}-\lambda)a_2+T_{23}a_3 &= 0,\\ T_{31}a_1+T_{32}a_2+(T_{33}-\lambda)a_3 &= 0.\end{aligned}$$

These are the same kind of equations that we discussed in Chapter 12 in connection with normal modes. (Compare (12.15).) As in that case, the equations are mutually consistent only if the determinant of the coefficients vanishes,

which would not have been reached on first reading.

$$\det(\mathbf{T}-\lambda\mathbf{1}) = \begin{vmatrix} T_{11}-\lambda & T_{12} & T_{13} \\ T_{21} & T_{22}-\lambda & T_{23} \\ T_{31} & T_{32} & T_{33}-\lambda \end{vmatrix} = 0.$$

When expanded, this determinant is a cubic equation for λ, whose three roots are either all real, or else one real and two complex conjugates of each other.

We shall now show that the latter possibility can be ruled out. For, suppose λ is a complex eigenvalue, and $\mathbf{a} = (a_1, a_2, a_3)$ the corresponding eigenvector, whose components may also be complex. We shall denote the complex conjugate eigenvalue by λ^*. Then, taking the complex conjugate of

$$\mathbf{T}\cdot\mathbf{a} = \lambda\mathbf{a},$$

we obtain

$$\mathbf{T}\cdot\mathbf{a}^* = \lambda^*\mathbf{a}^*,$$

where $\mathbf{a}^* = (a_x^*, a_y^*, a_z^*)$. Multiplying these two equations by $\mathbf{a}^*$ and $\mathbf{a}$ respectively, we obtain

$$\mathbf{a}^*\cdot\mathbf{T}\cdot\mathbf{a} = \lambda\mathbf{a}^*\cdot\mathbf{a},$$

$$\mathbf{a}\cdot\mathbf{T}\cdot\mathbf{a}^* = \lambda^*\mathbf{a}\cdot\mathbf{a}^*.$$

But since T is symmetric, the left sides are equal, by (C.7). Hence the right sides must be equal too, and since

$$\mathbf{a}^*\cdot\mathbf{a} = \mathbf{a}\cdot\mathbf{a}^* = \sum |a_i|^2 > 0,$$

this means that $\lambda^* = \lambda$. Thus the eigenvalue λ is real.

We have shown that the three eigenvalues λ_1, λ_2, λ_3 are all real, and we can find three real eigenvectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$.† (The subscripts no longer refer to the components; they distinguish the three vectors.) Next, we show that the eigenvectors corresponding to different eigenvalues are orthogonal. For, if

$$\mathbf{T}\cdot\mathbf{a}_1 = \lambda_1\mathbf{a}_1, \qquad \mathbf{T}\cdot\mathbf{a}_2 = \lambda_2\mathbf{a}_2,$$

then, multiplying the first equation by $\mathbf{a}_2$, and the second by $\mathbf{a}_1$, and again using the symmetry of $\mathbf{T}$, we obtain

$$\lambda_1\mathbf{a}_2\cdot\mathbf{a}_1 = \mathbf{a}_2\cdot\mathbf{T}\cdot\mathbf{a}_1 = \mathbf{a}_1\cdot\mathbf{T}\cdot\mathbf{a}_2 = \lambda_2\mathbf{a}_1\cdot\mathbf{a}_2.$$

Thus, either $\lambda_1 = \lambda_2$, or $\mathbf{a}_1\cdot\mathbf{a}_2 = 0$.

If the three eigenvalues are all distinct, then there are three orthogonal eigenvectors. We now show that, even if the eigenvalues are not distinct, one can find orthogonal eigenvectors. For, suppose that

$$\mathbf{T}\cdot\mathbf{a}_1 = \lambda\mathbf{a}_1, \qquad \mathbf{T}\cdot\mathbf{a}_2 = \lambda\mathbf{a}_2,$$

with the same eigenvalue λ. Then any vector $\mathbf{a}$ in the plane of $\mathbf{a}_1$ and $\mathbf{a}_2$ is also an eigenvector; for, if $\mathbf{a} = c_1\mathbf{a}_2 + c_2\mathbf{a}_2$, then

$$\mathbf{T}\cdot\mathbf{a} = c_1\lambda\mathbf{a}_1 + c_2\lambda\mathbf{a}_2 = \lambda\mathbf{a}.$$

We can always find a pair of orthogonal vectors in this plane; for example $\mathbf{a}_1$ and $\mathbf{a}_2' = \mathbf{a}_2 - \hat{\mathbf{a}}_1\hat{\mathbf{a}}_1\cdot\mathbf{a}_2$.

Finally, it is clear that any multiple of an eigenvector is also an eigenvector, so that we may choose our three eigenvectors to form an orthonormal triad $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. The symmetric tensor $\mathbf{T}$ is completely characterized by its eigenvalues, and the directions of its eigenvectors. If we choose our axes to be along the vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, then $\mathbf{T}$ must have a diagonal form,

$$\mathbf{T} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}. \tag{C.15}$$

For, if we multiply by $\mathbf{e}_1 = (1, 0, 0)$, we obtain a vector whose components are given by the first column of $\mathbf{T}$, and this must be equal to

† Actually, it is necessary to prove that, when the eigenvalue equation has a repeated root, one can find two (or three) independent eigenvectors. This is not true for an arbitrary tensor, but it is quite easy to show that it is for any symmetric tensor.

$\lambda_1\mathbf{e}_1$; and similarly for the other two columns. This is the result we used in Chapter 9 to reduce the inertia tensor to diagonal form. In that case, the eigenvectors are called principal axes, and the eigenvalues principal moments of inertia.

as referred to after the reference here in that Chapter at p 143

The procedure of finding normal co-ordinates for an oscillating system, discussed in §12.2, is quite similar. In that case, it is the potential energy function which is brought to 'diagonal' form.

Normal Modes §12-3 (12-20)

In conclusion, we note that in any co-ordinate system we may write $\mathbf{T}$ as a sum of dyadic products,

$$\mathbf{T} = \lambda_1\mathbf{e}_1\mathbf{e}_1+\lambda_2\mathbf{e}_2\mathbf{e}_2+\lambda_3\mathbf{e}_3\mathbf{e}_3. \qquad \text{(C.16)}$$

This is clearly true in the axes defined by $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. But, since the equation makes no reference to a choice of axes, it is true generally, It is easy to verify the eigenvalue equations directly by taking the dot product with $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ in turn.

PROBLEMS

1 Show that the rotation which takes the axes $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ into $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$ may be specified by $\mathbf{r}\to\mathbf{r}' = \mathbf{R}\cdot\mathbf{r}$, where the tensor $\mathbf{R}$ is $\mathbf{R} = \mathbf{i}'\mathbf{i}+\mathbf{j}'\mathbf{j}+\mathbf{k}'\mathbf{k}$. Write down the matrix of components of $\mathbf{R}$ if the rotation is through an angle θ about the z-axis. What is the tensor corresponding to the rotation which takes $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$ back into $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$? Show that $\tilde{\mathbf{R}}\cdot\mathbf{R} = \mathbf{1}$. (Such tensors are called *orthogonal*.)

2 The *trace* of a tensor $\mathbf{T}$ is the sum of its diagonal elements, $\mathrm{tr}(\mathbf{T}) = \Sigma_i T_{ii}$. Show that the trace is equal to the sum of the eigenvalues, and that the determinant is equal to the product of the eigenvalues.

3 The *double dot* product of two tensors is defined as $\mathbf{S}:\mathbf{T} = \mathrm{tr}(\mathbf{S}\cdot\mathbf{T}) = \Sigma_i\Sigma_j S_{ij}T_{ji}$. Evaluate $\mathbf{1}:\mathbf{1}$ and $\mathbf{1}:\mathbf{rr}$. Show that

$$(3\mathbf{r}'\mathbf{r}'-r'^2\mathbf{1}):(\mathbf{rr}-\tfrac{1}{3}r^2\mathbf{1}) = 3(\mathbf{r}'\cdot\mathbf{r})^2-r'^2r^2.$$

Hence show that the expansion (6.19) of the potential may be written

$$\phi(\mathbf{r}) = q\frac{1}{r}+\mathbf{d}\cdot\frac{\mathbf{r}}{r^3}+\tfrac{1}{2}\mathbf{Q}:\frac{\mathbf{rr}-\frac{1}{3}r^2\mathbf{1}}{r^5}+\cdots,$$

and write down the expression for the *quadrupole tensor* $\mathbf{Q}$. Show that $\mathrm{tr}(\mathbf{Q}) = 0$, and that in the axially symmetric case it has diagonal elements $-\tfrac{1}{2}Q$, $-\tfrac{1}{2}Q$, Q where Q is the quadrupole moment defined in Chapter 6. Show also that the gravitational quadrupole tensor is related to the inertia tensor by $\mathbf{Q} = \mathrm{tr}(\mathbf{I})\mathbf{1}-3\mathbf{I}$.

4 In an elastic solid in equilibrium, the force across a small area may have both a normal component (pressure or tension) and transverse components (shearing stress). Denote the ith component of force per unit area across an area with normal in the jth direction by T_{ij}. These are the components of the *stress tensor* $\mathbf{T}$. By considering the equilibrium of a small volume, show that the force across area A with normal in the direction of the unit vector $\mathbf{n}$ is $\mathbf{F} = \mathbf{T}\cdot\mathbf{n}A$. Show also by considering the equilibrium of a small rectangular volume that $\mathbf{T}$ is symmetric. What physical significance attaches to its eigenvectors?

Appendix D Units

The international standard units of length, mass and time are the *metre*, *kilogramme* and *second*. The kilogramme is defined by the standard international kilogramme kept at Sèvres in France. However, the other two basic units are defined by atomic standards, in terms of the wavelength of a designated spectral line of krypton and the period of a caesium line respectively.* These are three of the six fundamental units of the Système Internationale (SI). The others are the units of electric current (the *ampere*), temperature (the *kelvin*) and luminous intensity (the *candela*). Of these only the first is relevant here.

In this book we have generally used CGS units, based on the *centimetre*, *gramme* and *second*. The relation between these and the SI units is indicated in Table D.1. In the CGS system the electrostatic unit of charge (also called the *Franklin* (Fr)) is defined by saying that the force between a pair of unit charges separated by 1 cm is 1 dyne. Its dimensions are $M^{1/2}L^{3/2}T^{-1}$, so that the franklin may be written as 1 (erg cm)$^{1/2}$. If q is the charge in electrostatic units and c is the velocity of light then q/c (dimensions $M^{1/2}L^{1/2}$) is the charge in electromagnetic units. We have used the Gaussian system, in which electrical quantities are measured in electrostatic units and the magnetic field **B** in electromagnetic units (i.e. in *gauss* (G)).

In SI, the ampere is treated as an independent basic unit, although its definition in fact involves the first three units.† The corresponding unit of charge is the *coulomb*. SI also uses *rationalized* units, which means that factors of 4π appear in different places. Necessary changes to convert the formulae in this book to SI form are indicated in footnotes.

The basic units of the British and American systems now coincide and are defined in terms of the SI units. The *foot* and *pound* are precisely 0·3048 m and 0·453 592 37 kg, respectively. In the foot-pound-second system, the unit of force is the *poundal* ($\simeq$0·138 N) and of energy the *foot-poundal* ($\simeq$0·042 J).‡ It should be noted that the British and American systems still diverge in respect of certain

* The metre is 1 650 763·73 wavelengths in vacuum of the transition $2p_{10}-5d_5$ in ^{86}Kr. The second is 9 192 631 770 oscillation periods of the hyperfine transition between the levels $F=4$, $m_F=0$ and $F=3$, $m_F=0$ in the ground state of ^{133}Cs.

† It is defined in such a way that the constant μ_0 which has the dimensions of J A^{-2} m^{-1} (or H m^{-1}) is a pure number, $4\pi\times10^{-7}$. One ampere is equal to exactly one-tenth of the CGS electromagnetic unit of current.

‡ There is also in use a second system in which the unit of force is the *pound weight* and the unit of mass the *slug* ($\simeq$ 32·2 lb $\simeq$ 14·2 kg).

derived units such as the *gallon* and the *ton*. In this book 'ton' is always used to mean a metric ton, 1 t = 1000 kg.

TABLE D.1—CGS AND SI UNITS

*Dimension**	*Quantity*	*CGS unit*	*SI unit*	
L	length	centimetre (cm)	metre (m)	$= 10^2$ cm
M	mass	gramme (g)	kilogramme (kg)	$= 10^3$ g
T	time	second (s)	second (s)	= 1 s
MLT^{-2}	force	dyne (dyn)	newton (N)	$= 10^5$ dyn
ML^2T^{-2}	energy	erg	joule (J)	$= 10^7$ erg
ML^2T^{-3}	power	erg s^{-1}	watt (W)	$= 10^7$ erg s^{-1}
$ML^{-1}T^{-2}$	pressure†	microbar (μbar)	N m^{-2}	$= 10$ μbar
$M^{1/2}L^{3/2}T^{-1}$	charge	esu* (Fr)	coulomb (C)	$= 3\times10^9$ esu
$M^{1/2}L^{1/2}$		emu		$= 10^{-1}$ emu
$M^{1/2}L^{3/2}T^{-2}$	current	esu*	ampere (A)	$= 3\times10^9$ esu
$M^{1/2}L^{1/2}T^{-1}$		emu		$= 10^{-1}$ emu
$M^{1/2}L^{-1/2}T^{-1}$	electric field	esu*	volt m (Vm^{-1})	$= \frac{1}{3}\times10^{-4}$ esu
$M^{1/2}L^{-1/2}T^{-1}$	magnetic field	emu* (G)	tesla (T)	$= 10^4$ G

In general subdivisions and multiples of the basic units may be denoted by the following prefixes:

10^{-3} milli (m)	10^3 kilo (k)
10^{-6} micro (μ)	10^6 mega (M)
10^{-9} nano (n)	10^9 giga (G)
10^{-12} pico (p)	10^{12} tera (T)
10^{-15} femto (f)	
10^{-18} atto (a)	

* Gaussian system.

† In meteorology the usual unit of pressure is the millibar: 1 bar $= 10^6$ dyn cm^{-2} is roughly atmospheric pressure.

Bibliography

The following is a short list of selected textbooks which may be found helpful for further reading on classical mechanics and related subjects.

CORBEN, H. C. and STEHLE, P.: *Classical Mechanics*, 2nd ed., John Wiley & Sons, Inc., 1966.

GOLDSTEIN, H.: *Classical Mechanics*, Addison-Wesley Publishing Company, Inc., 1950.

JEFFREYS, Sir HAROLD: *The Earth*, 5th ed., Cambridge University Press, 1970.

LAMB, H.: *Hydrodynamics*, 6th ed., Cambridge University Press, 1932; reprinted by Dover Publications, Inc., 1945.

LANDAU, L. D. and LIFSHITZ, E. M.: *Mechanics*, 2nd ed., translated by J. B. Sykes and J. S. Bell, Pergamon Press, 1969.

MCCUSKEY, S. W.: *Introduction to Celestial Mechanics*, Addison-Wesley Publishing Company, Inc., 1963.

WHITTAKER, E. T.: *Analytical Dynamics*, 4th ed., Cambridge University Press, 1937; reprinted by Dover Publications, Inc., 1944.

Answers to Problems

Chapter 1

1 $m_1/m_2 = a_2/a_1$.
2 $m_A/m_B = 3$; $3\mathbf{v}/4$.

Chapter 2

2 0·96 s; 2π s^{-1}.
3 $x = -a$; $2\pi(2ma^3/c)^{1/2}$; (i) $|v| < (c/ma)^{1/2}$, (ii) $v < -(c/ma)^{1/2}$ or $(c/ma)^{1/2} < v < (2c/ma)^{1/2}$, (iii) $v > (2c/ma)^{1/2}$.
4 0·28 s; 0·18 cm.
5 2·01 s; 0·29:1:0·44.
8 $\omega_1 = (\gamma^2+\omega_0{}^2)^{1/2} \pm \gamma$.
10 3.
13 1·017.
14 $A_n = \text{constant}/n(1+n^2)$.

Chapter 3

1 (i) $a(x^2/2+yz)+b(xy^2+z^3/3)$;
(iii) $ar^2 \sin\theta \sin\varphi$; (vi) $\frac{1}{2}(\mathbf{a}\cdot\mathbf{r})^2$.

2 $\mathbf{F} = \dfrac{c}{r^3}\left(\dfrac{3z}{r^2}\mathbf{r}-\mathbf{k}\right)$; $\left(\dfrac{2c}{r^3}\cos\theta, \dfrac{c}{r^3}\sin\theta, 0\right)$.

3 4ω; $m\omega^2 l^4/r^3$.

5 (i) $\dfrac{m}{8\xi\eta}(\xi+\eta)(\dot{\xi}^2\eta+\dot{\eta}^2\xi)$;

(ii) $\dfrac{m}{2b^2}[(a^2+b^2)\dot{x}^2-2a\dot{x}\dot{\eta}+\dot{\eta}^2]$;

(iii) $\dfrac{m}{2}(\lambda^2+a^2\sin^2\theta)\left(\dot{\theta}^2+\dfrac{\dot{\lambda}^2}{a^2+\lambda^2}\right)$.

Chapter 4

1 42 000 km.
2 11·9 years.
3 162 cm s^{-2}, 2·35 km/s; 2580 cm s^{-2}, 60·4 km/s.
4 $\sqrt{2}v_E$; 45°; yes (unless one takes account of the perturbation due to Jupiter).
5 0·212 years.
6 $1{\cdot}32\times10^{-10}$.

7 $r^2\left[\left(1-\dfrac{kJ^2}{mE^2}\right)^{1/2}\cos 2(\theta-\theta_0)+1\right]=\dfrac{J^2}{mE}.$

8 3·9 days.

9 2·6 km/s; 2·8 km/s.

10 146 days; 55° behind earth.

11 $\dfrac{d\sigma}{d\Omega}=\dfrac{\pi^2k(\pi-\theta)}{mv^2\theta^2(2\pi-\theta)^2\sin\theta}.$

12 6·5 km/s; 60°.

13 $\omega=\left(-\dfrac{k}{ma^3}+\dfrac{c}{ma^5}\right)^{1/2},\ \omega'=\left(-\dfrac{k}{ma^3}-\dfrac{c}{ma^5}\right)^{1/2}.$

14 $1{\cdot}23\times10^{-5}$ cm.

15 $3{\cdot}6\times10^{-10}$ cm; 838 particles/s.

Chapter 5

1 1·001 76; right-hand (east) rail.

2 $1{\cdot}3\times10^{-2}$ mbar/km.

3 0·47 cm.

4 $B\ll4{\cdot}6\times10^9$ G; $1{\cdot}76\times10^{11}\ \mathrm{s}^{-1}$.

5 $5{\cdot}10\times10^{16}\ \mathrm{s}^{-1}$; $-3{\cdot}34\times10^{16}\ \mathrm{s}^{-1}$.

6 124 m.

8 11·3 cm E, $2{\cdot}8\times10^{-3}$ cm S.

Chapter 6

1 $\phi(z)=\dfrac{2q}{a^2}[(z^2+a^2)^{1/2}-|z|],\ \mathbf{E}(z)=\dfrac{2q\mathbf{k}}{a^2}\left(-\dfrac{z}{(z^2+a^2)^{1/2}}\pm1\right);$

$\mathbf{E}(z)\to\pm2\pi$ (surface charge density) $\mathbf{k}$; $-\frac{1}{2}qa^2$.

2 When $\mathbf{d}$ is in the same direction as $\mathbf{E}$.

3 $3q^2/5a$.

4 $(3\mathbf{d}\cdot\mathbf{r}\mathbf{r}-r^2\mathbf{d})/r^5$; (i) $\mathbf{F}=-\mathbf{F}'=-6dd'\mathbf{k}/r^4,\ \mathbf{G}=\mathbf{G}'=\mathbf{0}$;
(ii) $\mathbf{F}=-\mathbf{F}'=3dd'\mathbf{k}/r^4,\ \mathbf{G}=\mathbf{G}'=\mathbf{0}$;
(iii) $\mathbf{F}=-\mathbf{F}'=3dd'\mathbf{i}/r^4,\ \mathbf{G}=-dd'\mathbf{j}/r^3,\ \mathbf{G}'=-2dd'\mathbf{j}/r^3$;
(iv) $\mathbf{F}=\mathbf{F}'=\mathbf{0},\ \mathbf{G}=-\mathbf{G}'=-dd'\mathbf{k}/r^3$.

5 3500 tons weight cm^{-2}.

6 $6{\cdot}7\times10^6$ years.

7 4×10^{47} ergs.

8 1/9·6.

9 $\rho_1/\rho_2=2{\cdot}7$.

10 $-1{\cdot}45\times10^{-6}\ \mathrm{s}^{-1}$.

12 $a=(\pi k/4G)^{1/2}$.

Chapter 7

1 258 days.
2 $M_0/800$.
3 $x_1 = X+m_2x/M$, $x = X-m_1x/M$, where
$X = -\frac{1}{2}gt^2+m_1vt/M+X_0$, $x = l+(v/\omega)\sin\omega t$,
with $\omega = (k/\mu)^{1/2}$; $v = l\omega$.
4 $r_{\min} = 2{\cdot}41\ b$; $v_1 = 0{\cdot}383\ v$, $v_2 = 0{\cdot}663\ v$.
5 $m_1 = m_2$.
6 $12m_p$; 1/14.
7 $T^*/T = m_2/M$.
8 62·7°; 55°; 640 keV.
10 $$\frac{d\sigma}{d\Omega} = \left(\frac{2e^2}{mv^2}\right)^2 \cos\theta\left(\frac{1}{\sin^4\theta}+\frac{1}{\cos^4\theta}\right).$$
11 $6{\cdot}7\times10^3$ particles/s; same for both.
12 $$\frac{m_1}{M}\left[\left(\frac{m_2Q}{m_1}\right)^{1/2}\pm T^{1/2}\right]^2.$$
13 20.

Chapter 8

1 3·92 km/s; 144 kg.
2 (a) 13·8 km/s; (b) 16·6 km/s.
3 $\frac{1}{2}M_0u^2(1-e^{-v/u})$.
4 3 stages.
5 $-\rho Av^2$, $A = \pi r^2$.
7 $da/dt = dl/dt = -2\rho Ava/m$; 16 s, 13 km; 2·7 s, 2·2 km.
8 $v_1 = v/5$, $v_2 = v_3 = 2\sqrt{3}v/5$.
9 $R = a_0\omega_0{}^2/2\pi G\rho_0 = 6{\cdot}4\times10^7$ km.

Chapter 9

1 $4\sqrt{2}a/3$, $(2g/l)^{1/2}$; $-(Mg/8)\,(11+9\cos2\varphi,\ 9\sin2\varphi)$.
2 $1{\cdot}14\times10^3$ dynes.
3 69·3 rpm, 62 ergs.
4 $v/2a$; 1/3; $[3ga(\sqrt{2}-1)]^{1/2}$.
5 (a) 2·009 s; (b) 2·006 s.
6 $2{\cdot}7\times10^7$ years.
7 $I_3 = \frac{3}{10}Ma^2$, $I_1 = I_2 = \frac{3}{20}Ma^2+\frac{3}{5}Mh^2$; $h = a/2$;
$Z = 3h/4$; $I_3{}^* = I_3$, $I_1{}^* = I_2{}^* = \frac{51}{320}Ma^2$.
8 25·5 s; $1{\cdot}1\times10^{10}$ erg.
9 60°.

Chapter 10

1 $0{\cdot}244\ \mathrm{s}^{-1} = 2{\cdot}33$ rpm.
2 $0{\cdot}77$ s.
3 112 s.
4 $8{\cdot}8$ m.
6 (a) $16{\cdot}6\ \mathrm{s}^{-1}$; (b) $21{\cdot}6\ \mathrm{s}^{-1}$.
8 $8{\cdot}6\times 10^{-5}\ \mathrm{year}^{-1}$.
9 $22{\cdot}3$ s.

Chapter 11

1 $4{\cdot}33\ \mathrm{s}^{-1}$; $62{\cdot}6\ \mathrm{s}^{-1}$; $370\ \mathrm{s}^{-1}$.

2 $\left(\dfrac{4mgl}{(M+2m)\,a^2}\right)^{1/2}$.

4 $\dfrac{mg}{k}\left(\dfrac{M}{M+2m}\right)^2$.

6 $I\ddot{\varphi} = I_3\omega_3\Omega \sin\lambda \cos\varphi - I_1\Omega^2 \sin\lambda \sin\varphi \cos\varphi$; east and west.
7 I_1, I_3 are replaced by $I_1{}^* < I_1$, $I_3{}^* = I_3$.
9 $f(x) = 2xa/l$ for $-l/2 < x < l/2$,
$f(x) = 2(l-x)\,a/l$ for $l/2 < x < 3l/2$, etc.

Chapter 12

1 $A_X/A_x = 1, -m/M$; $2Ma/(M+m)$; no.
2 (a) $\omega_1 = (k/m)^{1/2}$, $\omega_2 = (3k/m)^{1/2}$;
(b) $\omega_1 = (k(l-a)/ml)^{1/2}$, $\omega_2 = (3k(l-a)/ml)^{1/2}$.
3 $x = a[\cos\omega_1 t \cos\omega_2 t + (1/\sqrt{2}) \sin\omega_1 t \sin\omega_2 t]$,
$y = a[2\cos\omega_1 t \cos\omega_2 t + (3/\sqrt{2}) \sin\omega_1 t \sin\omega_2 t]$,
where $\omega_{1,2} = \frac{1}{2}(\omega_+ \pm \omega_-)$ and $\omega_\pm = [(2\pm\sqrt{2})g/l]^{1/2}$.
4 $\alpha > k(3l/g)^{1/2}$.

5 $$\left(p^2 + \frac{\alpha+\alpha'}{2m}p + \frac{g}{l}\right)\left[p^2 + \left(\frac{\alpha+\alpha'}{2m} + 2\frac{\beta}{m}\right)p + \frac{g}{l} + \frac{2k}{m}\right] - \left(\frac{\alpha+\alpha'}{2m}\right)^2 p^2 = 0.$$

6 $a_{2r} = 0$, $a_{2r+1} = \left(\dfrac{l}{2}\right)^{1/2} \dfrac{8a}{(2r+1)^2\pi^2}$.

Chapter 13

1 $\dfrac{1}{\sin\alpha}\left(\dfrac{g}{r}\cos\alpha\right)^{1/2}$; 35·2°.

2 $(\omega^2-g^2/l^2\omega^2)^{1/2}$.

3 $y = (mg/k)(1-\cos\omega t)$, $x = x_0-\frac{1}{4}y$, $\omega = (4k/3m)^{1/2}$.

4 $\mathbf{J}^2 = p_\theta{}^2+p_\varphi{}^2/\sin^2\theta$.

5 $\omega_{3,\min}$ is reduced by a factor $I_1{}^*/I_1$.

7 $0<b<a$.

8 $\dot{\varphi} = J/m\rho^2$; $\dfrac{\partial U}{\partial\rho} = -\dfrac{J^2}{m\rho^3}-\dfrac{qJB_z}{cm\rho}$, $\dfrac{\partial U}{\partial z} = \dfrac{qJB_\rho}{cm\rho}$.

Appendix A

3 $\mathbf{a}(\nabla\cdot\mathbf{b})-\mathbf{b}(\nabla\cdot\mathbf{a})-(\mathbf{a}\cdot\nabla)\mathbf{b}+(\mathbf{b}\cdot\nabla)\mathbf{a}$.

5 $(\nabla^2\mathbf{A})_\rho = \nabla^2(A_\rho)-\dfrac{1}{\rho^2}\left(A_\rho+2\dfrac{\partial A_\varphi}{\partial\varphi}\right)$

$(\nabla^2\mathbf{A})_\varphi = \nabla^2(A_\varphi)-\dfrac{1}{\rho^2}\left(A_\varphi-2\dfrac{\partial A_\rho}{\partial\varphi}\right)$

$(\nabla^2\mathbf{A})_z = \nabla^2(A_z)$.

Appendix B

1 $\mathbf{B} = (J/cr^3)\,\mathrm{d}\mathbf{s}\wedge\mathbf{r}$; $\mathbf{F} = (JJ'/cr^3)\,\mathrm{d}\mathbf{s}\wedge(\mathrm{d}\mathbf{s}'\wedge\mathbf{r})$;
$\mathbf{F}+\mathbf{F}' = (JJ'/cr^3)\,\mathbf{r}\wedge(\mathrm{d}\mathbf{s}\wedge\mathrm{d}\mathbf{s}') \neq \mathbf{0}$.

Appendix C

1 $R = \begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}$

3 $\mathbb{Q} = \iiint \rho(\mathbf{r}')\,(3\mathbf{r}'\,\mathbf{r}'-r'^2\mathbf{1})\,\mathrm{d}^3\mathbf{r}'$.

Index

Printed by William Clowes & Sons Limited, London, Colchester and Beccles

Physical constants

Velocity of light

$$c = 2{\cdot}998 \times 10^{10}\ \mathrm{cm\ s^{-1}} = 2{\cdot}998 \times 10^{8}\ \mathrm{m\ s^{-1}}$$

Gravitational constant

$$G = 6{\cdot}67 \times 10^{-8}\ \mathrm{dyn\ cm^2\ g^{-2}} = 6{\cdot}67 \times 10^{-11}\ \mathrm{N\ m^2\ kg^{-2}}$$

Mass of hydrogen atom

$$m_{\mathrm{H}} = 1{\cdot}673 \times 10^{-24}\ \mathrm{g} = 1{\cdot}673 \times 10^{-27}\ \mathrm{kg}$$

Mass of electron

$$m_e = 9{\cdot}109 \times 10^{-28}\ \mathrm{g} = 9{\cdot}109 \times 10^{-31}\ \mathrm{kg}$$

Charge of electron

$$e = 4{\cdot}803 \times 10^{-10}\ \mathrm{esu} = 1{\cdot}602 \times 10^{-19}\ \mathrm{C}$$

$$e/c = 1{\cdot}602 \times 10^{-20}\ \mathrm{emu}$$

Defined values

Standard gravitational acceleration

$$g_n = 980{\cdot}665\ \mathrm{cm\ s^{-2}} = 9{\cdot}806\,65\ \mathrm{m\ s^{-2}}$$

Normal atmospheric pressure

$$1\ \mathrm{atm} = 1013{\cdot}25\ \mathrm{mbar} = 1{\cdot}013\,25 \times 10^{5}\ \mathrm{N\ m^{-2}}$$

Properties of earth

Mass	$M = 5{\cdot}98 \times 10^{27}\ \mathrm{g} = 5{\cdot}98 \times 10^{24}\ \mathrm{kg}$
Radius (polar)	$R_p = 6357\ \mathrm{km}$
(equatorial)	$R_e = 6378\ \mathrm{km}$
(mean)*	$R = 6371\ \mathrm{km}$
Semi-major axis of orbit	$a = 1{\cdot}496 \times 10^{8}\ \mathrm{km}$
Eccentricity of orbit	$e = 0{\cdot}0167$
Orbital period (year)	$\tau = 3{\cdot}156 \times 10^{7}\ \mathrm{s}$
Mean orbital velocity	$v = 29{\cdot}8\ \mathrm{km\ s^{-1}}$
Surface escape velocity	$v_e = 11{\cdot}2\ \mathrm{km\ s^{-1}}$
Mass of sun	$M_S = 1{\cdot}99 \times 10^{33}\ \mathrm{g} = 3{\cdot}32 \times 10^{5}\ M$
Mass of moon	$M_M = 7{\cdot}35 \times 10^{25}\ \mathrm{g} = 0{\cdot}0123\ M$
Semi-major axis of lunar orbit	$a_M = 3{\cdot}84 \times 10^{5}\ \mathrm{km}$

* Defined as the radius of a sphere with volume equal to that of the earth.

Classical Mechanics, Second Edition

is a textbook intended for use in the first or second year of an Honours degree course in physics. Classical mechanics is treated as a branch of physics, rather than as applied mathematics, and the emphasis of the book is on the basic principles involved, in particular those aspects which are of importance in more modern branches of physics such as quantum mechanics, nuclear physics, and relativity.

The first half of the book is devoted to the basic principles of the subject and the mechanics of a single particle, while the second half deals with the mechanics of systems of particles and rigid bodies. The crucial role of the conservation laws is particularly stressed. The Lagrangian method is introduced in the third chapter and developed in later chapters. The last chapter contains a discussion of the Hamiltonian method, stressing the relationship between symmetries and conservation laws.

In this second edition, some sections have been substantially revised, and minor errors corrected. It also contains answers to problems, and quotes equations in their SI forms as well as in Gaussian units.

T W B Kibble

obtained an MA in Mathematics and Natural Philosophy, a BSc in Physics, and a PhD in Mathematical Physics from Edinburgh University. In 1958 he went to the California Institute of Technology as a Commonwealth Fund Fellow. He returned to England a year later and has been at Imperial College, London, ever since, except for one year, 1967-68, which he spent as a visitor at the University of Rochester, Rochester, NY. He became Professor of Theoretical Physics at Imperial College in 1970. His research interests are quantum field theory, particularly quantum electrodynamics, and theory of elementary particles.

McGraw-Hill Book Company (UK) Limited

MAIDENHEAD · BERKSHIRE · ENGLAND

07 084019 9